On Inverse Network Problems and their Generalizations

Çiğdem Güler

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Gutachter: Prof. Dr. Horst W. Hamacher
Prof. Dr. Rainer E. Burkard

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No man is an island, entire of itself; every man is a piece of the continent, a part of the main. If a clod be washed away by the sea, Europe is the less, as well as if a promontory were, as well as if a manor of thy friend's or of thine own were. Any man's death diminishes me because I am involved in mankind; and therefore never send to know for whom the bell tolls; it tolls for thee.

John Donne (1572-1631)

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Contents

Introduction and Outline

Introduction

In the past few decades, optimization problems with estimated problem parameters have drawn considerable attention from researchers. For this kind of problems one often knows a priori an optimal solution based on observations or experiments, but is interested in finding a set of parameters, such that the known solution is optimum (i) and the deviation from the initial estimates is minimized (ii). The problem of recalculating the parameters satisfying (i) and (ii) is known as *inverse optimization problem*.

Ahuja and Orlin (2001) mention, in their paper, that the major application area for inverse optimization is geophysical sciences and it were, indeed, geophysicists to first study such problems. At the beginning of 90's, a well-known study by Burton and Toint (1992, 1994) attracted the interest of mathematicians to this topic. In their papers, the authors study inverse shortest path problems to predict the movements of earthquakes. Since then inverse versions of several optimization problems have been intensely investigated in diverse areas of application such as:

- Medical imaging in X-ray tomography where a CT-scan of a body part is exploited to estimate its dimensions given other information on the body
- Imposing tolls in transportation networks, in order to enforce the use of system equilibrium routes instead of user optimal ones (Dial, 1999, 2000)
- High-speed Asynchronous Transfer Mode (ATM) networks to obtain reliable and self-configuring systems (Faragó *et al.*, 2003)

Among the optimization problems studied in the context of inverse optimization, the combinatorial problems such as network flows are the most popular ones to analyze their inverses. Especially inverse versions of maximum flow and minimum cost flow problems have thoroughly been investigated in the literature. In these network flow problems there are two important problem parameters: flow capacities of the arcs of the network and costs incurred by sending a unit flow on these arcs. Capacity changes for maximum flow problems and cost changes for minimum cost flow problems have been studied under several distance measures such as rectilinear (ℓ_1), Chebyshev (ℓ_∞), and Hamming distances. These norms are usually the most preferred ones because of their handiness and practical relevance.

In this thesis, we also deal with inverse network flow problems and their counterparts tension problems under the aforementioned distance measures. Our major goals are

- to enrich the inverse optimization theory by introducing new inverse network problems that have not yet been treated in the literature,
- to extend the well-known combinatorial results of inverse network flows for more general classes of problems with inherent combinatorial properties such as matroid flows on regular matroids and monotropic programming.

To accomplish our first objective, we analyzed the inverse maximum flow problem under ℓ_∞-norm, which, to the best of our knowledge, has not yet been covered in the literature. Besides, we introduced the *capacity inverse minimum cost flow problem*, in which only arc capacities are perturbed. In this way, we attempted to close the gap between the capacity perturbing inverse network problems and the cost perturbing ones.

Among the distance measures mentioned earlier, rectilinear (ℓ_1) and Chebyshev (ℓ_∞) distances are emphasized explicitly throughout the thesis, whereas Hamming distance is, from time to time, employed as a second objective to minimize the number of perturbations. The reason for such a choice lies in the existence of several optimal solutions for some inverse problems, for which, we believe, selecting the one with the minimum number of perturbations is practically appropriate.

The foremost purpose of studying inverse tension problems on networks was to achieve a well-established generalization of the inverse network problems. Since tensions are duals of network flows, carrying the theoretical results of network flows over to tensions follows quite intuitively. We made use of this intuitive link between network flows and tensions to gain more insight into the inverse network optimization problems and to gradually build up a generalization to matroid flows and monotropic programs.

At this point it is necessary to mention that this is not the first study on the generalization of inverse network problems. Cai *et al.* (1999) consider the inverse polymatroidal flow problem under asymmetric weighted ℓ_1-norm where the amount of modification allowed is restricted. This problem can be formulated as a combinatorial linear program, which can be transformed to another linear program whose dual can be interpreted as a minimum cost circulation problem on a related network. Later the same authors extended these results for the inverse problems of submodular functions on digraphs (Cai *et al.*, 2000). Moreover, Yang (1998) studied inverse submodular function problems under Chebyshev norm. He also employed linear programming duality to solve this inverse optimization problem in strongly polynomial time.

Recall that matroids are special cases of polymatroids and polymatroids are equivalent to submodular functions (Edmonds, 1970; Fujishige, 1991). Hence, the results on these general inverse problems are also valid for matroids. However, these general results on inverse polymatroid and submodular function problems exploit linear programming formulations and duality. On the other hand, our objective is to generalize the combinatorial results of the inverse network flows, which exploit cycles. Consequently, the existence of special combinatorial structures such as cycles and cuts of a graph is essential for our generalizations. Since matroids and monotropic programs preserve these combinatorial structures, analyzing the inverse problems of matroid flows and monotropic problems is not void although the results on inverse polymatroids and submodular functions on digraphs can be carried over.

In the parts of the thesis dealing with network flows and tensions, we did not only concentrate on analyzing the mathematical formulations of the inverse problems but also presented the necessary algorithms to solve them along with the complexity results. On the other hand, we did not endeavor to develop generalized algorithms for solving inverse problems of matroid flows and monotropic programs and restricted our research to elaborate theoretical results concerning the mathematical formulations of these problems. A study on generalized algorithms would go far beyond the scope of this thesis, and hence, is left for further research.

The following sections of this thesis are in large parts based upon work published or submitted for publication elsewhere, or include collaborative work:

1. Section 2.2 is based upon "Capacity Inverse Minimum Cost Flow Problem", by Ç. Güler and H.W. Hamacher, accepted for publication in "Journal of Combinatorial Optimization" and online since 30 April 2008.

2. Parts of Chapter 3 and Chapter 5 are based upon "Inverse Tension Problems and Monotropic Optimization", by Ç. Güler, submitted to "Journal of Combinatorial Optimization".

Outline and Main Results

This section is intended to give an overview of this thesis and provide an outline of how the presentation of the material is organized.

In Chapter 1, an introduction to the basic terminology and properties of network flow problems and their inverse versions is given. Moreover, we provide a survey on the most important facts and results of Inverse Optimization, which will be needed in the latter analysis.

As it was already mentioned, one of the main goals of this thesis is to enhance the theory of inverse combinatorial optimization by investigating new inverse network problems that have not yet been treated in the literature. Chapter 2 intends to fulfill this objective by studying the inverse maximum flow problem under Chebyshev norm and by introducing the capacity inverse minimum cost flow problem. In this chapter we show that the inverse maximum flow problem under ℓ_∞-norm can be solved in strongly polynomial time by finding a maximum capacity path on the residual graph with respect to the given feasible flow. Furthermore, we analyze a lexicographic version of this problem, in which we minimize the number of affected arcs among the optimum solutions of Chebyshev norm. We present a strongly polynomial algorithm for this problem, as well, which is a slightly modified version of the algorithm in Zhang and Liu (2006).

In Chapter 2 we also introduce a new class of inverse network flow problems which has so far not been treated in the literature. This problem is called *capacity inverse minimum cost flow problem* and attempts to model the inverse minimum cost flow problems in which only the arc flow capacities are changed. Under rectilinear norm we prove that the problem is $\mathcal{NP}$-hard by a reduction from the *feedback arc set problem*, which is known to be $\mathcal{NP}$-hard (Garey and Johnson, 1979; Karp, 1972). Under Chebyshev distance the problem can be solved in strongly polynomial time by a greedy algorithm. Similar to the inverse maximum flows, we investigated the bicriteria version of this problem, as well. Since the problem is $\mathcal{NP}$-hard under rectilinear norm, so is the bicriteria problem. Therefore, we introduce a 2-phase approximation algorithm, which we call Bicriteria Approximation Algorithm and provide computational results in Section 2.2.4.

Chapter 3 deals with inverse tension problems and extends the combinatorial results of inverse network flow problems for tensions. We show in this chapter that the duality relation between tensions and flows is valid for their respective inverse problems, as well. Furthermore, studying inverse tension problems on networks helps to intensify the understanding of the combinatorial characteristics of the inverse network problems and serves as a bridge to accomplish a well-established generalization of these problems.

The second goal of this thesis is to extend the well-known combinatorial results of inverse network flows to more general classes of problems with inherent combinatorial properties such as matroid flows on regular matroids and monotropic programming. Therefore, Chapters 4 and 5 define and investigate the inverse problems of matroid flows and monotropic programs.

In Chapter 4, we study inverse maximal M-flow and minimum cost M-flow problems. The inverse maximal M-flow problem under rectilinear norm is equivalent to solving a maximal M-flow problem on an auxiliary regular matroid, which can be defined using the given admissible M-flow. Under Chebyshev norm, the problem can be formulated as identifying an augmenting circuit on the incremental matroid, which maximizes the capacity of the minimum capacity element on it. The cost inverse minimum cost M-flow problem is also analyzed under rectilinear and Chebyshev distances. Under the rectilinear norm the problem can be solved by determining a minimum cost collection of disjoint residual circuits in the incremental matroid, whereas it is sufficient to identify a minimum mean residual circuit to solve the problem under Chebyshev norm. Chapter 4 provides a generalization of the capacity inverse minimum cost flow problem to matroid flows under Chebyshev norm, as well. Since the flow case is $\mathcal{NP}$-hard under rectilinear norm, we left the generalization of this problem to matroid flows for future research.

Monotropic programming deals with optimization problems that minimize a separable convex function subject to linear constraints. In Chapter 5, we consider the cost inverse primal problem with separable linear costs under ℓ_1 and ℓ_∞ norms. We show that the combinatorial results of inverse (ordinary) network flow problems can be extended to these monotropic programs. We also analyze the generalized minimum cost flow problem as an example of monotropic programs, which do not possess totally unimodularity.

Finally, Chapter 6 concludes this thesis by discussing further research topics in this area.

Far better an approximate answer to the right question, which is often vague, than an exact answer to the wrong question, which can always be made precise.

John W. Tukey (1962)

1

Preliminaries

The purpose of this first chapter is to establish the language, fix the terminology, and summarize the most important mathematical concepts used throughout this thesis. Section 1.1 contains the terminology and basic results from the field of Network Optimization where a special emphasis is put on network flow problems and their optimality conditions. For an extensive introduction on network flows, reference is made to standard texbooks as, for instance, Ahuja *et al.* (1993). In Section 1.2, a brief introduction is given to inverse network flow problems and other related problems together with well-known results in this field.

1.1 Network Flow Problems

In this section we review some basics of the network flow problems whose inverse versions are the starting point for this thesis. We assume that the reader is acquainted with the basic definitions of Graph Theory. Throughout this thesis we employ the terms *network* and *graph* interchangeably.

Let $G = (N, A)$ be a connected directed graph with a node set N of n nodes and an arc set A of m arcs. Each arc $(i, j) \in A$ has an associated cost c_{ij} that denotes the cost per unit flow on that arc. It is assumed that the flow cost varies linearly with the amount of flow. With each arc on the graph a maximum flow capacity $u_{ij} \in \mathbb{R}$ (*upper flow capacity/bound*) and a minimum flow amount $l_{ij} \in \mathbb{R}$ (*lower flow capacity/bound*) with $l_{ij} \leq u_{ij}$ are also associated. The nodes of the graph have supplies or demands of value $b(i) \in \mathbb{R}$. These supplies and demands satisfy $\sum_{i=1}^{n} b(i) = 0$. The well-known

Algorithm	Running Time
Capacity scaling algorithm	$O((m \log U)(m + n \log n))$
Cost scaling algorithm	$O(n^3 \log(nC))$
Double scaling algorithm	$O(nm \log U \log(nC))$
Minimum mean cycle-canceling algorithm	$O(n^2 m^3 \log n)$
Repeated capacity scaling algorithm	$O((m^2 \log n)(m + n \log n))$
Enhanced capacity scaling algorithm	$O((m \log n)(m + n \log n))$

Table 1.1: Minimum cost flow algorithms and their time bounds ($C = \max_{(i,j)\in A} c_{ij}$)

linear programming (LP) formulation of the ***minimum cost flow problem*** is

$$\min \quad \sum_{(i,j)\in A} c_{ij} x_{ij} \tag{1.1a}$$

subject to

$$\sum_{j\in N^+(i)} x_{ij} - \sum_{j\in N^-(i)} x_{ji} = b(i) \qquad \forall\, i \in N \tag{1.1b}$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad \forall\, (i,j) \in A \tag{1.1c}$$

Here, $N^+(i)$ and $N^-(i)$ are the sets of nodes adjacent from and to node i, respectively. The constraints in (1.1b) are usually referred to, in the literature, as *flow conservation* or *mass balance constraints*, whereas (1.1c) are called *capacity* or *flow bound constraints*. If all demands and supplies are equal to zero, then the flow is called a *circulation*.

In ***maximum flow problem***, the flow incurs no costs and the aim is to find a feasible solution that sends the maximum amount of flow from a specified *source node* s to another specified *sink node* t. The LP formulation of the maximum flow problem is

$$\max \quad v \tag{1.2}$$

subject to

$$\sum_{i\in N^+(s)} x_{si} = v$$

$$\sum_{j\in N^+(i)} x_{ij} - \sum_{j\in N^-(i)} x_{ji} = 0 \qquad \forall\, i \in N \backslash \{s,t\}$$

$$\sum_{i\in N^-(t)} x_{it} = -v$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad \forall\, (i,j) \in A$$

It should be noted that the maximum flow problem can be formulated as a minimum cost flow problem by introducing an additional arc (t,s) to the graph G with cost $c_{ts} = -1$ and flow bounds $(l_{ts}, u_{ts}) = (-\infty, \infty)$, and by setting $x_{ts} = v$.

In the literature both maximum flow and minimum cost flow problems are known

Algorithm	**Running Time**
Labeling algorithm	$O(nmU)$
Capacity scaling algorithm	$O(nm \log U))$
Successive shortest path algorithm	$O(n^2m)$
Generic preflow-push algorithm	$O(n^2m)$
FIFO preflow-push algorithm	$O(n^3)$
Highest-label preflow-push algorithm	$O(n^2\sqrt{m})$
Excess scaling algorithm	$O(nm + n^2 logU)$

Table 1.2: Maximum flow algorithms and their running times ($U = \max_{(i,j)\in A} u_{ij}$)

to be solvable in strongly polynomial time. Here we do not analyze the algorithms, which solve the maximum flow and minimum cost flow problems. We only provide the names of these algorithms and their running times in tables 1.1 and 1.2 and refer to Ahuja *et al.* (1993) for further details. Currently, the best available time bound for the minimum cost flow problem is

$$O(\min\{nm \log(n^2/m) \log(nC), nm(\log \log U) \log(nC), (m \log m)(m + n \log n)\}).$$

The three bounds in this expression are, respectively, due to Goldberg and Tarjan (1990), Ahuja *et al.* (1992), and Orlin (1988).

1.1.1 Optimality for Maximum Flows

In the literature there are two different characterizations of the optimality conditions for maximum flow problems. The first one is obtained from flow augmenting paths while the other one stems from $s - t$ cuts.

Definition 1.1. A *flow augmenting path* with respect to a given feasible solution $\hat{x}$ of a maximum flow problem is a path P in $G = (N, A)$ from s to t such that

$$\hat{x}_{ij} < u_{ij} \quad \forall\, (i,j) \in P^+ \quad \text{and} \quad \hat{x}_{ij} > l_{ij} \quad \forall\, (i,j) \in P^-.$$

where P^+ and P^- denote the set of forward and backward arcs of the path P, respectively.

Theorem 1.2. ***(Flow Augmenting Paths Theorem)*** *A feasible flow $\hat{x}$ on $G = (N, A)$ is a maximum flow if and only if there does not exist a flow augmenting path with respect to $\hat{x}$.*

Definition 1.3. An $s - t$ *cut* on $G = (N, A)$ is a cut $\omega = (S, \bar{S})$ with $s \in S$ and $t \in \bar{S}$ where $S, \bar{S} \subseteq N$ and $\bar{S} = N \backslash S$.

We denote the set of forward arcs of an $s - t$ cut as ω^+, i.e. $(i,j) \in A$ with $i \in S$ and $j \in \bar{S}$, and the set of backward arcs as ω^-, i.e. $(i,j) \in A$ with $i \in \bar{S}$ and $j \in S$.

Then, the capacity of an $s-t$ cut is defined as

$$u(\omega) = \sum_{(i,j)\in\omega^+} u_{ij} - \sum_{(i,j)\in\omega^-} l_{ij}.$$

Theorem 1.4. ***(Max-Flow Min-Cut Theorem)*** *The maximum value of the flow from a source node s to a sink node t in a capacitated network is equal to the minimum capacity among all $s-t$ cuts.*

1.1.2 Optimality for Minimum Cost Flows

The concept of ***residual graphs*** plays a crucial role in network flow optimization problems. In defining a residual graph with respect to a given flow $\hat{x}$, we use the following basic idea. Suppose that the arc (i,j) carries a flow of $\hat{x}_{ij}$, then we can send an additional flow of $u_{ij} - \hat{x}_{ij}$ from node i to node j along the arc (i,j) without violating the capacity condition. Moreover, we can send up to $\hat{x}_{ij}$ units of flow from node j to node i, which is equivalent to canceling the existing flow on the arc (i,j). Using this intuitive idea, the residual graph with respect to a given flow $\hat{x}$ can be defined as follows:

Definition 1.5. The *residual graph* with respect to a given flow $\hat{x}$, denoted by $G(\hat{x},u,l)$, is a directed graph with the node set N and the arc set

$$A(\hat{x}) = \{(i,j) \in A : \hat{x}_{ij} < u_{ij}\} \cup \{(j,i) : (i,j) \in A \text{ and } \hat{x}_{ij} > l_{ij}\}.$$

The arcs $(i,j) \in A(\hat{x})\backslash A$ have costs of $-c_{ji}$ and upper capacities of $r_{ij} = \hat{x}_{ji} - l_{ji}$, whereas the respective parameters for the arcs $(i,j) \in A(\hat{x}) \cap A$ are c_{ij} and $r_{ij} = u_{ij} - \hat{x}_{ij}$. All the arcs in $G(\hat{x},u,l)$ have lower flow capacities of 0 (see Figure 1.1).

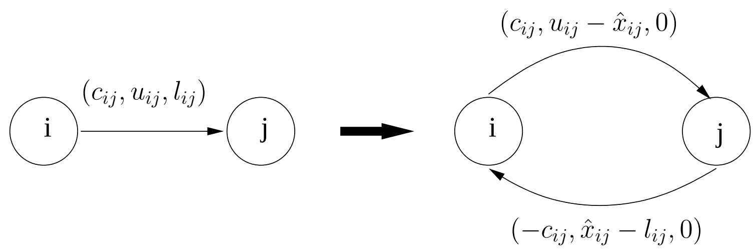

Figure 1.1: Construction of the residual graph with respect to a given flow $\hat{x}$

In the literature it is well-known that the residual graph with respect to an optimum flow of the minimum cost flow problem possesses a certain property which helps to characterize the optimality. We call this property, following Ahuja *et al.* (1993), Negative Cycle Property.

Property 1.6. ***(Negative Cycle Property)*** *A feasible flow $\hat{x}$ to a minimum cost flow problem is an optimal flow if and only if the corresponding residual graph $G(\hat{x}, u, l)$ does not contain any negative (cost) directed cycles.*

Another necessary and sufficient condition of the optimality for minimum cost flow problems exploits residual graphs together with ***reduced costs***. Suppose that we associate a real number $\pi(i) \in \mathbb{R}$ with each node $i \in N$, and refer to $\pi(i)$ as the ***potential*** of node i. For a given set of node potentials π, the *reduced cost* of an arc (i, j) is equal to $c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)$.

Theorem 1.7. ***(Reduced Cost Optimality Conditions)*** *A feasible solution $\hat{x}$ is an optimal solution of the minimum cost flow problem if and only if some set of node potentials $\pi(i)$ for all $i \in N$ satisfy the following reduced cost optimality conditions*

$$c_{ij}^{\pi} \geq 0 \quad \forall \ (i, j) \in A(\hat{x}).$$

Theorem 1.7 can also be restated in terms of the original graph instead of the residual graph.

Theorem 1.8. ***(Complementary Slackness Optimality Conditions)*** *A feasible solution $\hat{x}$ is an optimal solution of the minimum cost flow problem if and only if for some set of node potentials $\pi(i)$ for all $i \in N$, the reduced costs and flow values satisfy*

$$\text{If} \quad \begin{array}{l} c_{ij}^{\pi} > 0, \\ l_{ij} < \hat{x}_{ij} < u_{ij}, \\ c_{ij}^{\pi} < 0, \end{array} \quad \text{then} \quad \begin{array}{l} \hat{x}_{ij} = l_{ij}, \\ c_{ij}^{\pi} = 0, \\ \hat{x}_{ij} = u_{ij}. \end{array}$$

1.2 Literature on Inverse Network Flows

While solving an optimization problem, we mostly assume that the problem parameters such as costs, capacities, etc. are known and try to find an optimum solution according to these given parameters. However, in practice, it is possible that we only know the estimates for these problem parameters, but can observe the optimum solutions of the problem through experiments. The main idea of *inverse optimization* is to find new estimates of the parameters, which differ from the initial estimates as little as possible, such that the observed solutions are optimum with respect to the new estimates. In terms of this definition, a typical optimization problem is a *forward problem* because it identifies the values of the observable problems parameters, i.e., optimal decision variables, given the values of the model parameters such as cost coefficients and capacities. On the other hand, an inverse optimization problem is a *backward* *problem*, which consists of computing the values of the model parameters given the values of the observable parameters.

In the past few decades there has been an increasing interest in inverse optimization problems in the context of mathematical optimization, and a variety of inverse optimization problems have been studied by researchers. Heuberger (2004) published a first thorough survey on inverse combinatorial optimization problems. In his survey he describes several classes of inverse problems in detail and reviews solution methods proposed in the literature. Here, we provide only a literature review for those inverse problems that are of special interest for us and refer to the survey paper by Heuberger (2004) for a detailed analysis of the existing literature on inverse optimization problems.

1.2.1 Inverse Linear Programming Problem

This problem has first been investigated by Zhang and Liu (1996, 1999). The authors formulate the inverse linear programming problem as a new linear problem and derive LP formulations for several inverse network flow problems. Huang and Liu (1999) and Ahuja and Orlin (2001) achieve the same result and show that the inverse problem of a linear program is also a linear program. Since the approach provided by Ahuja and Orlin (2001) is rather general, we briefly sketch their approach and results.

Consider the following linear program

$$\text{Minimize} \quad \sum_{j \in J} c_j x_j \tag{1.3a}$$

subject to

$$\sum_{j \in J} a_{ij} x_j \geq b(i) \quad \forall\, i \in I, \tag{1.3b}$$

$$l_j \leq x_j \leq u_j \quad \forall\, j \in J, \tag{1.3c}$$

where J denotes the index set of the decision vector x and I denotes the index set of the constraints I. Here, l_j and u_j represent the lower and upper bounds on x_j, respectively.

Let $\hat{x}$ be a feasible solution to the LP. The aim of the inverse linear programming problem is to perturb the cost vector from c to $\hat{c}$ such that the given feasible solution $\hat{x}$ will be an optimum solution with respect to the linear program with the cost vector $\hat{c}$, denoted by LP($\hat{c}$), and the amount of perturbation is minimum according to some distance measure. We call $\hat{c}$ *inverse feasible* with respect to $\hat{x}$ if $\hat{x}$ is an optimum solution of LP($\hat{c}$) and *inverse optimum* if $\hat{c}$ is inverse feasible and $\|c - \hat{c}\|$ is minimum.

Let B denote the index set of binding constraints in (1.3b) with respect to $\hat{x}$, i.e.

$B = \{i \in I \ : \ \sum_{j\in J} a_{ij}\hat{x}_j = b(i)\}$. Moreover, we define

$$L = \{j \in J \ : \ \hat{x}_j = l_j\} \quad \text{and} \quad U = \{j \in J \ : \ \hat{x}_j = u_j\}.$$

Then, by using the complementary slackness conditions (Hamacher and Klamroth, 2001) of dual linear programs, we can formulate the inverse linear programming problem under weighted ℓ_1-norm as

$$\text{Minimize} \quad \sum_{j\in J} w_j|c_j - \hat{c}_j| \tag{1.4a}$$

subject to

$$\sum_{i\in B} a_{ij}\pi(i) + \lambda_j = \hat{c}_j \quad \forall\, j \in L, \tag{1.4b}$$

$$\sum_{i\in B} a_{ij}\pi(i) - \varphi_j = \hat{c}_j \quad \forall\, j \in U, \tag{1.4c}$$

$$\sum_{i\in B} a_{ij}\pi(i) = \hat{c}_j \quad \forall\, j \in F, \tag{1.4d}$$

$$\pi(i) \geq 0 \quad \forall\, i \in B, \quad \lambda_j \geq 0 \quad \forall\, j \in L,$$
$$\varphi_j \geq 0 \quad \forall\, j \in U, \tag{1.4e}$$

where $F = \{j \in J \ : \ l_j < \hat{x}_j < u_j\}$, and $\pi(i)$, λ_j, and φ_j are the dual variables corresponding to the constraints of the LP (1.3b, 1.3c). w_j denote the positive weight values with respect to the cost modification for decision variable x_j for all $j \in J$.

It is well-known that minimizing $|c_j - \hat{c}_j|$ is equivalent to minimizing $\alpha_j + \beta_j$ subject to $c_j - \hat{c}_j = \alpha_j - \beta_j$ where $\alpha_j, \beta_j \geq 0$. Using this transformation, the formulation of the inverse linear programming problem (1.4a - 1.4e) can be converted into a linear program.

Under the weighted ℓ_∞-norm, the objective function of the above formulation (1.4a) should be replaced by

$$\text{Minimize} \quad \max_{j\in J} \{w_j|c_j - \hat{c}_j|\}. \tag{1.5}$$

Analogously, the inverse linear programming problem under weighted ℓ_∞-norm can be converted into a linear program after eliminating the absolute value signs and the maximization of the terms. In order to eliminate the maximization, we need to introduce a nonnegative variable θ, and add the constraints $w_j\alpha_j + w_j\beta_j \leq \theta$ for each $j \in J$.

Ahuja and Orlin (2001) dualize the LP formulation of the inverse linear programming problem and come up with the following two LPs for the inverse linear programming problem under weighted ℓ_1 and ℓ_∞-norms, respectively. Recall that here $\hat{x}$ is a part of the given data and x is a decision variable.

$$\text{(LP-Rectilinear)} \qquad \text{Minimize} \quad \sum_{j \in J} c_j x_j \tag{1.6a}$$

subject to

$$\sum_{j \in J} a_{ij} x_j \geq b(i) \quad \forall\, i \in B, \tag{1.6b}$$

$$u_j - w_j \leq x_j \leq u_j \quad \forall\, j \in U, \tag{1.6c}$$

$$l_j \leq x_j \leq l_j + w_j \quad \forall\, j \in L, \tag{1.6d}$$

$$\hat{x}_j - w_j \leq x_j \leq \hat{x}_j + w_j \quad \forall\, j \in F. \tag{1.6e}$$

$$\text{(LP-Chebyshev)} \qquad \text{Minimize} \quad \sum_{j \in J} c_j x_j \tag{1.7a}$$

subject to

$$\sum_{j \in J} a_{ij} x_j \geq b(i) \quad \forall\, i \in B, \tag{1.7b}$$

$$x_j \geq \hat{x}_j \quad \forall\, j \in L, \tag{1.7c}$$

$$x_j \leq \hat{x}_j \quad \forall\, j \in U, \tag{1.7d}$$

$$\sum_{j \in J} \frac{1}{w_j} |x_j - \hat{x}_j| \leq 1 \quad \forall\, j \in F. \tag{1.7e}$$

Theorem 1.9. *(Ahuja and Orlin, 2001) Let $\hat{x}$ be a feasible solution to the LP (1.3a - 1.3c) and $\pi(i)$ be the value of the dual variable associated with the i^{th} constraint of (1.6b) in an optimal solution to the dual of the LP (1.6a - 1.6e). We define $c_j^{\pi} := c_j - \sum_{i \in B} a_{ij} \pi(i)$. Then, an optimal solution to the inverse linear programming problem under weighted ℓ_1- norm is given by*

$$\hat{c}_j = \begin{cases} c_j - |c_j^{\pi}| & \text{if } c_j^{\pi} > 0 \text{ and } \hat{x}_j > l_j, \\ c_j + |c_j^{\pi}| & \text{if } c_j^{\pi} < 0 \text{ and } \hat{x}_j < u_j, \\ c_j & \text{otherwise.} \end{cases} \tag{1.8}$$

Theorem 1.10. *(Ahuja and Orlin, 2001) Let $\hat{x}$ be a feasible solution to the LP (1.3a - 1.3c), $\pi(i)$ be the value of the dual variable associated with the i^{th} constraint of (1.7b) in an optimal solution to the dual of the LP (1.7a - 1.7e). Then, an optimal solution to the inverse linear programming problem under weighted ℓ_∞- norm is given by (1.8).*

Sokkalingam (1995) develops a duality theory for inverse linear programming problems using duality results in convex analysis. His results generalize the theorems 1.9 and 1.10.

1.2.2 Inverse Maximum Flow and Minimum Cut Problems

Inverse maximum flow problems can be defined as follows: Given a feasible flow $\hat{x}$ to a maximum flow problem, we try to modify the arc capacities as little as possible according to some norm such that the given feasible flow is a maximum flow. Yang *et al.* (1997) study this problem under ℓ_1-norm for a maximum flow problem with only upper flow capacities, i.e. $l_{ij} = 0$ for all arcs $(i, j) \in A$. Their aim is to perturb the upper flow capacities as little as possible from u to $\hat{u}$ such that $\hat{x}$ will be a maximum flow with respect to the new upper flow capacities $\hat{u}$. Yang *et al.* (1997) additionally limit the maximum amount of modification by setting upper and lower bounds on the new upper flow capacities, i.e., $u_{ij} - \alpha_{ij} \leq \hat{u}_{ij} \leq u_{ij} + \delta_{ij}$ for all $(i, j) \in A$ and $\delta_{ij}, \alpha_{ij} \geq 0$.

Yang *et al.* (1997) exploit maximum flow - minimum cut duality in order to characterize necessary and sufficient conditions for the feasibility and optimality of the inverse problem. They show that the inverse maximum flow problem under ℓ_1-norm can be transformed into a maximum flow problem on an auxiliary digraph with at most $3|A|$ arcs.

Deaconu (2008) extends the results of Yang *et al.* (1997) for maximum flow problems with upper and lower flow capacities on the arcs, i.e., there exists $(i, j) \in A$ with $l_{ij} \neq 0$, and show that the inverse problem, which is perturbing both the upper and lower flow capacities, can be solved as a maximum flow problem on a similar auxiliary digraph as that of Yang *et al.* (1997).

Zhang and Liu (2006) analyze the inverse maximum flow problem under Hamming distance where they consider $l_{ij} = 0$ for all $(i, j) \in A$. The Hamming distance between the given upper flow capacities u_{ij} and the modified upper capacities $\hat{u}_{ij}$ is defined as $H(u_{ij}, \hat{u}_{ij}) = 0$ if $\hat{u}_{ij} = u_{ij}$ and 1 otherwise. Under the weighted sum type Hamming distance the objective function is to minimize $\sum_{(i,j)\in A} w_{ij} H(u_{ij}, \hat{u}_{ij})$ whereas the objective function of weighted bottleneck type Hamming distance is to minimize $\max_{(i,j)\in A}\{w_{ij} H(u_{ij}, \hat{u}_{ij})\}$. Zhang and Liu (2006) present strongly polynomial algorithms to solve the inverse maximum flow problem under Hamming distance. These algorithms generate an auxiliary graph using the residual graph and solve a minimum $s - t$ cut problem to identify the upper flow capacities that are inverse optimum.

Given a directed graph $G = (N, A)$ with source s and sink t, and capacities $u : A \to \mathbb{R}_+$ and $l : A \to 0$, the minimum $s - t$ cut problem is finding an $s - t$ cut ω with the minimum capacity $u(\omega)$. In the inverse version of this problem, the aim is to determine the minimum modification of the capacities u to $\hat{u} : A \to \mathbb{R}_+$ according to a given distance measure such that a given $s - t$ cut ω is a minimum $s - t$ cut for the graph G with the new capacities $\hat{u}$, denoted by $G(\hat{u})$.

Yang *et al.* (1997) study the inverse minimum cut problem under unit weight ℓ_1-norm and show that this problem can be transformed into a maximum flow problem in $G' = (N, A\backslash\omega^-)$. The authors also consider the case where the capacity modifications are constrained and show that the constrained case can be solved as a minimum cost flow problem in an auxiliary graph with at most $2|A|$ arcs.

Ahuja and Orlin (2001) use a linear programming approach to reduce the inverse minimum cut problem under weighted ℓ_1-norm into solving a minimum cost flow problem. In a later paper, Ahuja and Orlin (2002) exploit combinatorial arguments to achieve the same result.

Zhang and Cai (1998) consider an inverse minimum cut problem where a set of $s-t$ cuts is given. The capacities of the arcs should be modified so that all the given $s-t$ cuts are minimum cuts with respect to the new capacities. The authors show that the inverse minimum cut problem with multiple $s-t$ cuts also leads to a minimum cost flow problem in an auxiliary graph with at most $2|A|$ arcs under asymmetric weighted ℓ_1-norm, i.e. there exist 2 weight values $w_{ij}^+ \in \mathbb{R}_+$ and $w_{ij}^- \in \mathbb{R}_+$ associated with the perturbation of each arc $(i, j) \in A$.

The inverse minimum cut problem under weighted ℓ_∞-norm with integer capacities and weights is studied by Ahuja and Orlin (2002). They use combinatorial arguments and binary search to solve the problem in polynomial time.

Shigeno (2002) analyzes the inverse minimum cut problem under ℓ_∞-norm in 2 different versions. In the first version, there exist lower capacities l_{ij} on arcs which can also be modified along with the upper capacities. In the second version the lower capacities are given fixed values that cannot be perturbed. For both of the versions Shigeno (2002) provides polynomial algorithms by exploiting the fact that the inverse minimum cut problems are closely related to maximum mean cut problems. His solution approach uses a parametric search for maximum mean-cut problems.

Yang (2001) considers the inverse minimum cut problem with a partially given solution and shows that this problem is $\mathcal{NP}$-hard.

1.2.3 Inverse Minimum Cost Flow Problem

Given a feasible flow $\hat{x}$ to a minimum cost flow problem, the aim of the inverse minimum cost flow problem is to modify the cost vector as little as possible according to some norm such that the given feasible flow $\hat{x}$ is a minimum cost flow.

Zhang and Liu (1996) and Ahuja and Orlin (2001) use the linear programming approach, which was described previously, to analyze the inverse minimum cost flow problem under unit weight ℓ_1-norm. In this case, the inverse problem reduces to a minimum cost circulation problem on the residual graph corresponding to $\hat{x}$ with unit capacities. For the weighted ℓ_∞-norm the inverse problem turns out to be solvable as a minimum cost-to-time ratio cycle problem. The same results are also obtained by Sokkalingam (1995) by using his duality theory on inverse linear programming problems.

Sokkalingam (1995) studies the inverse minimum cost flow problem under unit weight ℓ_2-norm, and transforms this problem into a quadratic cost flow problem.

In another paper Ahuja and Orlin (2002) analyze the combinatorial aspects of inverse minimum cost flow problem under unit weight ℓ_1 and ℓ_∞ norms. They show that the optimum objective function value is, for the former problem, equal to the minimum cost of a collection of arc-disjoint cycles in the residual graph with respect to $\hat{x}$, whereas the latter problem can be reduced to finding a minimum mean cost cycle in the residual graph.

Dial (1999, 2000) studies the problem of computing minimal-revenue tolls in a road network. The problem is to impose tolls in such a way that the paths chosen by the users coincide with the system optimal paths and the total amount of tolls raised is minimum. Dial (1999, 2000) formulates the problem as an inverse minimum cost flow problem and exploits implicitly the linear programming to solve the problem.

1.2.4 Some Further Results on Inverse Optimization Problems

Cai *et al.* (1999) consider the inverse polymatroidal flow problem under asymmetric weighted ℓ_1-norm where the amount of modification allowed is restricted. This problem can be formulated as a combinatorial linear program and further transformed into another linear program whose dual can be interpreted as a minimum cost circulation problem on a related network. Later, the same authors extend these results for inverse problems of submodular functions on digraphs (Cai *et al.*, 2000). Moreover, Yang (1998) studies inverse submodular function problems under Chebyshev norm where he adopts the linear programming approach to solve the problem. It turns out that many combinatorial optimization problems such as network flow problems can be interpreted as special cases of submodular function maximization models. Therefore, the results of Cai *et al.* (1999, 2000) and Yang (1998) can be carried over to inverse network flow problems.

Zhang and Liu (2002) propose an optimization model for general inverse optimization problems and show that most of the combinatorial problems can be fit into this model as special cases. They also suggest a Newton-type algorithm for their model under ℓ_∞-norm.

The cowboys have a way of trussing up a steer or a pugnacious bronco which fixes the brute so that it can neither move nor think. This is the hog-tie, and it is what Euclid did to geometry.

Eric T. Bell (1883 - 1960)

2 Inverse Network Flows with Capacity Change

In the literature capacity modifications were examined, in particular, for minimum cut and maximum flow problems (see Section 1.2). To the best of our knowledge, there do not exist any studies on the inverse maximum flow problem (abbreviated as IMaxF) under ℓ_∞-norm. In Section 2.1, we close this gap and study the inverse maximum flow problem under ℓ_∞-norm.

Another class of inverse network flow problems which has so far not been treated in the literature, but seems to have some potential in applications is the *capacity inverse minimum cost flow problem*, in which only the arc flow capacities are changed. In Section 2.2, we consider this problem under rectilinear and Chebyshev norms and provide complexity results. Moreover, we propose a heuristic for the bicriteria problem, where we minimize, among all optimal solutions of the Chebyshev norm, the number of affected arcs. In Section 2.2.4 we discuss the results of computational experiments for the proposed heuristic. This part of the thesis was published in Güler and Hamacher (2008).

2.1 Inverse Maximum Flow Problem under ℓ_∞ Norm

In this section we will analyze the inverse maximum flow problem under ℓ_∞-norm on a digraph $G = (N, A)$ with a node set N of n nodes and an arc set A of m arcs. There exist lower and upper bounds for the flows on arcs which we denote by $l : A \to \mathbb{R}^m$ and $u : A \to \mathbb{R}^m$, respectively, and these bounds satisfy $l_{ij} \leq u_{ij}$ for all $(i,j) \in A$. Given a nonoptimal feasible flow $\hat{x} : A \to \mathbb{R}^m$ and a weight function $w : A \to \mathbb{R}^m_+$, the inverse maximum flow problem under ℓ_∞-norm can be formulated as changing the lower and upper bounds such that $\hat{x}$ will be a maximum flow for the maximum flow problem with the new bounds $\hat{l}$ and $\hat{u}$, and

$$\max_{(i,j)\in A} w_{ij}(\max\{|\hat{l}_{ij} - l_{ij}|, |\hat{u}_{ij} - u_{ij}|\})$$

is minimum.

By the max-flow min-cut theorem (Theorem 1.4) a maximum flow x yields a saturated $s-t$ cut ω, i.e. with $x_{ij} = u_{ij}$ for all $(i,j) \in \omega^+$ and $x_{ij} = l_{ij}$ for all $(i,j) \in \omega^-$. Let Ω denote the set of all $s-t$ cuts in G. Since in our case $\hat{x}$ is not a maximum flow, all $s-t$ cuts are unsaturated. That is, for all $s-t$ cuts $\omega \in \Omega$, there exists some $(i,j) \in \omega^+$ such that $\hat{x}_{ij} < u_{ij}$ or some $(i,j) \in \omega^-$ such that $\hat{x}_{ij} > l_{ij}$. Consequently, we can reformulate our inverse problem as follows:

Lemma 2.1. *The inverse maximum flow problem under ℓ_∞-norm is equivalent to finding an $s-t$ cut ω in G such that*

$$c_\omega = \max\{\max_{(i,j)\in\omega^+} w_{ij}(u_{ij} - \hat{x}_{ij}), \max_{(i,j)\in\omega^-} w_{ij}(\hat{x}_{ij} - l_{ij})\} \tag{2.1}$$

is minimum. In particular, it suffices to change the upper bounds for the outgoing arcs of the cut and the lower bounds for the incoming arcs.

In order to solve (2.1), we define the residual graph $G(\hat{x}) = (N, A(\hat{x}))$ with

$$A(\hat{x}) = (A\backslash\{(i,j) : \hat{x}_{ij} = u_{ij}\}) \cup \{(j,i) : (i,j) \in A \text{ and } \hat{x}_{ij} > l_{ij}\}$$

and assign a capacity function $c : A(\hat{x}) \to \mathbb{R}^{|A(\hat{x})|}$ with

$$c_{ij} = \begin{cases} w_{ij}(u_{ij} - \hat{x}_{ij}) & \text{for } (i,j) \in A \\ w_{ji}(\hat{x}_{ji} - l_{ji}) & \text{for } (i,j) \in A(\hat{x})\backslash A. \end{cases} \tag{2.2}$$

Note that if $\hat{x}$ is a maximum flow, then there exists an $s-t$ cut $\omega(\hat{x})$ in $G(\hat{x})$ such that $\omega(\hat{x})^+ = \emptyset$.

Lemma 2.2. *Let $\Omega(\hat{x})$ denote the set of all $s-t$ cuts in $G(\hat{x})$. The objective function value of inverse maximum flow problem under ℓ_∞-norm is equal to*

$$c^* = \min_{\omega(\hat{x})\in\Omega(\hat{x})} \max_{(i,j)\in\omega(\hat{x})^+} c_{ij}. \tag{2.3}$$

Proof: By the construction of $G(\hat{x})$, for each $s-t$ cut ω in G there exists an $s-t$ cut $\omega(\hat{x})$ in $G(\hat{x})$ given by

$$\omega(\hat{x})^+ = \{(i,j) : (i,j) \in \omega^+ \text{ with } \hat{x}_{ij} < u_{ij}\} \cup \{(j,i) : (i,j) \in \omega^- \text{ with } \hat{x}_{ij} > l_{ij}\}.$$

Thus, $c^* = \min_{\omega\in\Omega} c_\omega$, which is, by Lemma 2.1, equal to the objective function of the inverse maximum flow problem under ℓ_∞-norm. ■

Next, we will show that the inverse maximum flow problem under Chebyshev norm can be solved by solving a maximum capacity path problem. The capacity of a directed $s-t$ path P on a graph G is the minimum of the capacities of the arcs in P. Then, the *maximum capacity path problem* (or *bottleneck shortest path problem*) is finding a directed $s-t$ path of maximum capacity (Schrijver, 2003). In order to reduce the inverse problem into a maximum capacity problem we exploit the bottleneck min-max duality which was proved by Fulkerson (1966), and extended by Hamacher (1981) to regular matroids.

Theorem 2.3. ***(Fulkerson, 1966)*** *Let $G = (N, A)$ be a digraph with $s, t \in N$, and let $c : A \to \mathbb{R}^{|A|}$ be a capacity function. Then,*

$$\max_{P\in\mathcal{P}} \min_{(i,j)\in P} c_{ij} = \min_{\omega\in\Omega} \max_{(i,j)\in\omega} c_{ij} \tag{2.4}$$

where $\mathcal{P}$ and Ω are the sets of all $s-t$ paths and cuts in G, respectively.

Edmonds and Fulkerson (1970) generalized this result to clutters. Let E be a finite set. A *family* $\mathfrak{F}$ on E is a family of subsets of E and a *clutter* $\mathfrak{R}$ on E is a family on E such that no member of $\mathfrak{R}$ is contained in another member of $\mathfrak{R}$.

Theorem 2.4. ***(Edmonds and Fulkerson, 1970)*** *For any clutter $\mathfrak{R}$ on a finite set E, there exists a unique clutter $\mathfrak{S} = b(\mathfrak{R})$ on E such that, for any function f from E to $\mathbb{R}$,*

$$\min_{R\in\mathfrak{R}} \max_{x\in R} f(x) = \max_{S\in\mathfrak{S}} \min_{x\in S} f(x). \tag{2.5}$$

Specifically, $\mathfrak{S}$ is the clutter consisting of the minimal subsets of E that have nonempty intersection with every member of $\mathfrak{R}$.

Any pair of families $\mathfrak{R}$ and $\mathfrak{S}$ on E is called a *blocking system* on E if they satisfy (2.5) for every f and regardless of whether they are clutters. Edmonds and Fulkerson (1970) proved that any blocking system fulfills the following property.

Property 2.5. *For any partition of E into two sets E_0 and E_1 ($E_0 \cap E_1 = \emptyset$ and $E_0 \cup E_1 = E$), either a member of $\mathfrak{R}$ is contained in E_0 or a member of $\mathfrak{S}$ is contained in E_1, but not both.*

They also showed that any pair of families on E fulfilling this property forms a blocking system. In addition, they proved that the pair $\mathfrak{S} = b(\mathfrak{R})$ specified in Theorem 2.4 is the one and only clutter on E having Property 2.5. Hence, by using Property 2.5 we can derive the following conclusion, which was also mentioned by Hamacher (1976).

Corollary 2.6. *Let $G = (N, A)$ be a digraph with $s, t \in N$, and let $c : A \to \mathbb{R}^{|A|}$ be a capacity function. Then,*

$$\max_{P \in \mathcal{P}} \min_{(i,j) \in P} c_{ij} = \min_{\omega \in \Omega} \max_{(i,j) \in \omega^+} c_{ij} \tag{2.6}$$

where $\mathcal{P}$ is the set of all elementary ***directed*** *$s - t$ paths, Ω is the set of all $s - t$ cuts in G, and ω^+ denote the forward arcs of the cut $\omega \in \Omega$.*

Proof: Let $\mathfrak{R}$ be the set of all elementary directed $s - t$ paths. Moreover, we denote the sets of the forward arcs of all $s - t$ cuts in G with $\mathfrak{S}$. Now, we need to show the validity of Property 2.5 for $\mathfrak{R}$ and $\mathfrak{S}$.

Consider the capacity function $c : A \to \{0, 1\}$. We define

$$E_0 = \{(i,j) \in A : c_{ij} = 0\} \quad \text{and} \quad E_1 = \{(i,j) \in A : c_{ij} = 1\}.$$

If the maximum flow from s to t is equal to 1, then there exists an elementary directed path P with $P \subseteq E_1$. By the max-flow min-cut theorem (Theorem 1.4), the minimum capacity $s - t$ cut has a forward arc of capacity 1, which means that there does not exist $\omega^+ \in \mathfrak{S}$ such that $\omega^+ \subseteq E_0$. Similarly if the maximum flow from s to t equals 0, then there exists $\omega^+ \in \mathfrak{S}$ with $\omega^+ \subseteq E_0$ but there does not exist P with $P \subseteq E_1$. Hence, Property 2.5 holds for $\mathfrak{R}$ and $\mathfrak{S}$, and they form a blocking system. Therefore, (2.5) holds for $\mathfrak{R}$ and $\mathfrak{S}$.

■

The main conclusion for inverse maximum flow problems under ℓ_∞-norm can be derived from Lemma 2.2 and Corollary 2.6.

Theorem 2.7. *The optimum objective function value of the inverse maximum flow problem under ℓ_∞-norm on a digraph G with respect to a given feasible flow $\hat{x}$ can be calculated by solving a maximum capacity (elementary) path problem on the residual graph $G(\hat{x})$ with respect to the capacities defined by equations (2.2).*

The maximum capacity path problem is a well-known combinatorial problem, which has several real-life applications (Listrovoi and Khrin, 1998; Fernandez *et al.*, 1998). The problem can be solved in $O(m + n \log n)$ time by modifying Dijkstra's algorithm and using Fibonacci heaps (Schrijver, 2003). Gabow (1985) employed binary search to solve the problem in $O(m \log_n C)$ time where $C = \|c\|_\infty$ with c nonnegative integer. Punnen (1996) showed that if a bottleneck combinatorial optimization problem of size m with ordered weights can be solved in $O(\xi(m))$ time, then the problem with arbitrary weights can be solved in $O(\xi(m) \log^*(m))$ time, where $\log^* m$ is the iterated logarithm of m. Thus, the maximum capacity path problem can be solved in $O(m \log^* m)$ time. More recently, Kaibel and Peinhardt (2006) proposed an algorithm of $O(m \log \log m)$ running time for the directed graphs with integer arc capacities. For a brief survey of bottleneck network flow problems, we refer to Hamacher (1976) and

Punnen and Zhang (2007) where a generalized algorithm for the bottleneck network flow problems is provided, as well.

Here we present the Labeling Algorithm, which is a modification of Dijkstra's. The validity proof of the algorithm follows analogous to the proof of the classical Dijkstra's algorithm.

Algorithm 1. (Labeling Algorithm - Modified Dijkstra's)

1. Set Label$(s) := \infty$ and all other nodes in N to 0. Also assign the set of to be processed nodes with $N^* := N$.

2. If $N^* = \emptyset$, STOP.
   Else, choose a node $i \in N^*$ with the maximum Label(i) and for all outgoing arcs (i, j) assign
   $$\text{Label}(j) := \max\{\min\{\text{Label}(i), c_{ij}\}, \text{Label}(j)\}. \tag{2.7}$$
   If Label$(j) = \min\{$Label$(i), c_{ij}\}$, then set Predecessor$(j) := i$.

3. Set $N^* := N^* \backslash \{i\}$.

Theorem 2.7 yields, of course, only the optimal objective function value of the inverse maximum flow problem under ℓ_∞-norm. However, once we have this optimum objective function value, we can easily identify an optimum solution. Suppose that c^* is the optimum objective function value, then we set for each arc $(i, j) \in A$,

- $u^*_{ij} = \hat{x}_{ij}$ if $w_{ij}(u_{ij} - \hat{x}_{ij}) \leq c^*$ and $u^*_{ij} = u_{ij}$ otherwise,
- $l^*_{ij} = \hat{x}_{ij}$ if $w_{ij}(\hat{x}_{ij} - l_{ij}) \leq c^*$ and $l^*_{ij} = l_{ij}$ otherwise.

It is easy to verify that the pair of lower and upper bound vectors (l^*, u^*) generated in this way is an optimal solution to the inverse maximum flow problem under ℓ_∞-norm.

Note that if we determine an optimum solution in this way, we might have to modify both lower and upper bounds for some arcs. However, by Lemma 2.1 we know that there exists an optimum solution to the inverse problem where for each arc either the upper bound or the lower bound has to be perturbed. In order to find this solution, we need to find an $s - t$ cut on $G(\hat{x})$ satisfying Lemma 2.2. This can be achieved by applying the Minimum Capacity Cut Algorithm of Christofides (1975) (Algorithm 2), which was also provided in the article of Listrovoi and Khrin (1998). This algorithm determines an $s - t$ cut that minimizes the capacity of its maximum capacity arc.

Algorithm 2. (Minimum Capacity Cut Algorithm)

Input: The residual graph $G(\hat{x}) = (N, A(\hat{x}))$ with capacity $c : A(\hat{x}) \to \mathbb{R}^{|A(\hat{x})|}$ defined with equation (2.2)

Output: An $s-t$ cut $\omega(\hat{x}) = (S, \bar{S})$ on graph $G(\hat{x})$ satifying Lemma 2.2

1. Start with $s-t$ cut $\bar{K}(\{s\}, N\backslash\{s\})$ on $G(\hat{x})$ and find the maximum capacity $\bar{c}$ of the forward arcs of $\bar{K}$.
2. Construct the spanning subgraph $G^* = (N, A^*)$ of $G(\hat{x})$ with $A^* = \{(i,j) \in A(\hat{x}) : c_{ij} \geq \bar{c}\}$.
3. Find the set of reachable nodes $R^*(s)$ from s on the subgraph G^*.
4. If $t \in R^*(s)$, then $c^* = \bar{c}$ and any $s-t$ cut in the spanning subgraph G^* has the maximum capacity c^*. If $t \notin R^*(s)$, go to Step 5.
5. Define $\bar{K}$ as the cut $(R^*(s), N\backslash R^*(s))$ and find the maximum capacity of the arcs in the new cut. Go to Step 2.

The worst case running time of Minimum Capacity Cut Algorithm is $O(mn)$, which is slower than the Labeling Algorithm (Algorithm 1) with Fibonacci heaps. Hence, if it is not compulsory to find an optimum $s-t$ cut, it would be more appropriate to use the Labeling Algorithm for solving the inverse problem.

2.1.1 Bicriteria Inverse Maximum Flow Problem

An extension of the inverse maximum flow problem under Chebyshev norm is a lexicographic bicriteria problem where we minimize the number of perturbations among all the optimum solutions. In this case, the second objective is a unit weight sum-type Hamming distance, i.e.,

$$\min \sum_{(i,j)\in A} (H(u_{ij}, \hat{u}_{ij}) + H(l_{ij}, \hat{l}_{ij})), \tag{2.8}$$

where $H(a, \hat{a}) = 0$ if $\hat{a} = a$ and $H(a, \hat{a}) = 1$ otherwise. Zhang and Liu (2006) showed that the inverse maximum flow problem under weighted sum-type Hamming distance is equivalent to solving a minimum $s-t$ cut problem. We can propose a similar approach (Algorithm 3) in order to solve the bicriteria inverse maximum flow problem in strongly polynomial time.

Algorithm 3. (Bicriteria Inverse Max Flow Algorithm)

Input: The residual graph $G(\hat{x}) = (N, A(\hat{x}))$ with capacity $c : A(\hat{x}) \rightarrow \mathbb{R}^{|A(\hat{x})|}$ defined with equation (2.2)

Output: An $s-t$ cut $\omega(\hat{x}) = (S, \bar{S})$ on graph $G(\hat{x})$ having the minimum number of forward arcs and satifying Lemma 2.2

1. Find the optimum objective function value c^* of inverse maximum flow problem under ℓ_∞-norm by solving a maximum path problem on graph $G(\hat{x})$.

2. Assign a new capacity function $c' : A(\hat{x}) \rightarrow \mathbb{R}^{|A(\hat{x})|}$ for all $(i,j) \in A(\hat{x})$ such that

$$c'_{ij} = \begin{cases} 1 & \text{if} \quad c_{ij} \leq c^* \\ \left(\frac{n^2}{4} + 1\right) & \text{if} \quad c_{ij} > c^* \end{cases} \tag{2.9}$$

3. Find a minimum $s - t$ cut ω on $G(\hat{x})$ with the capacity function c'.

Because this algorithm is a slightly modified version of the algorithm in Zhang and Liu (2006), we refer to their article for a correctness proof. The worst case running time of the algorithm is $\mathcal{O}(n^3)$ since the most costly operation is identifying the minimum $s - t$ cut in the last step (Ahuja *et al.*, 1993).

Once we identify a minimum $s - t$ cut ω on $G(\hat{x})$, we can generate an optimum solution (l^*, u^*) to the bicriteria inverse maximum flow problem on graph G by assigning

$$l^*_{ij} = \begin{cases} \hat{x}_{ij} & \text{if} \quad (j,i) \in \omega^+ \\ l_{ij} & \text{otherwise} \end{cases} \qquad u^*_{ij} = \begin{cases} \hat{x}_{ij} & \text{if} \quad (i,j) \in \omega^+ \\ u_{ij} & \text{otherwise} \end{cases} \tag{2.10}$$

2.1.2 Extension to Zero Lower Bounds Case $(l_{ij} = 0)$

In this section, we will extend the previous results of inverse maximum flow problem under Chebyshev norm to the case where $l_{ij} = 0$ for all $(i,j) \in A$ and only upper bounds u_{ij} can be perturbed.

Again by max-flow min-cut theorem (Theorem 1.4) and by the fact that $\hat{x}$ is not a maximum flow, we know that all the $s - t$ cuts are unsaturated.

Lemma 2.8. *Let $\bar{\Omega} = \{\omega \in \Omega : \hat{x}_{ij} = 0, \quad \forall\ (i,j) \in \omega^-\}$. An inverse maximum flow problem under Chebyshev norm has a feasible solution if and only if $\bar{\Omega} \neq \emptyset$.*

Lemma 2.9. *The inverse maximum flow problem under ℓ_∞-norm is equivalent to finding an $s - t$ cut $\omega \in \bar{\Omega}$ such that*

$$c_\omega = \max_{(i,j) \in \omega^+} w_{ij}(u_{ij} - \hat{x}_{ij})$$

is minimum.

We again define the residual graph $G(\hat{x}) = (N, A(\hat{x}))$ with respect to $\hat{x}$ and assign a capacity function $c : A(\hat{x}) \rightarrow \mathbb{R}^{|A(\hat{x})|}$ with

$$c_{ij} = \begin{cases} w_{ij}(u_{ij} - \hat{x}_{ij}) & \text{for } (i,j) \in A \\ M & \text{for } (i,j) \in A(\hat{x}) \backslash A \end{cases} \tag{2.11}$$

where M is sufficiently large.

Corollary 2.10. *If the optimum objective function value of the maximum capacity (directed) path problem on the residual graph $G(\hat{x})$ with the arc capacities defined by (2.11) is equal* *to M, then the inverse maximum flow problem under ℓ_∞-norm is infeasible. Otherwise, the optimum objective function value of the inverse maximum flow problem is equal to the optimum objective function value of the maximum capacity path problem on the digraph $G(\hat{x})$.*

2.2 Capacity Inverse Minimum Cost Flow Problem

Minimum cost flow problems have already been considered in the context of inverse optimization by several authors (Section 1.2). To the best of our knowledge, until now only changes of the cost function have been considered for the minimum cost flows. In the *(cost) inverse minimum cost flow problem*, the initial cost vector c is replaced by $\hat{c}$ such that a given feasible network flow $\hat{x}$ is optimal with respect to cost $\hat{c}$. Our goal is to close the gap between capacity perturbing inverse optimization problems and the cost perturbing ones by analyzing inverse minimum cost flow problems, in which only the arc capacities are changed. We call this problem *capacity inverse minimum cost flow problem* in order to distinguish it from the inverse minimum cost flow problem where the cost vector is changed. In general, we use denotations IMCF$_u$ and IMCF$_c$ for the capacity and cost inverse cases, respectively, but mostly drop the capacity index subsequently, since in this section we only deal with this class of problems.

The practical motivation to work on IMCF$_u$ arises from radiation therapy planning where we would like to find a therapy plan that minimizes the underdosage of cancerous tissue and the overdosage of healthy organs (Hamacher and Küfer, 2002). In order to model the radiation problem as a network flow problem, we interpret the voxels as arcs of some graph. The flows on these arcs represent the doses deposited in the voxels where lower and upper bounds on the dosage are modeled as lower and upper capacities. Then, minimizing the overdosage and underdosage on the voxels is nothing but minimizing the change in the arc capacities. Due to the large number of voxels, the resulting network model of the radiation problem is very large and complex. The original study, which has lead to this network flow model, can be found in Güler (2006).

2.2.1 Problem Definition

Given a digraph $G = (N, A)$ with a node set N of n nodes and an arc set A of m arcs with arc flow capacities $u : A \to \mathbb{Z}_+^m$ and a feasible integer flow $\hat{x}$ for an instance of a minimum cost flow problem, the capacity inverse minimum cost flow problem is

$$\min \quad \|u - \hat{u}\| \tag{2.12a}$$

subject to

$$\hat{x}_{ij} \leq \hat{u}_{ij} \quad \forall\, (i,j) \in A \tag{2.12b}$$

$\hat{x}$ is an optimal min cost flow (2.12c)

with respect to capacity $\hat{u}$

Note that $\hat{x}$ and u are part of the data while $\hat{u}$ is the vector of variables to be determined.

In order to model this inverse problem mathematically, it is first necessary to replace the verbal formulation (2.12c) by a formal one. To achieve this, we use the Negative Cycle Property (Property 1.6), which is a well-known optimality condition of minimum cost flow problems (Ahuja *et al.*, 1993).

As a consequence of the Negative Cycle Property, we know that the residual graph $G(\hat{x}, u)$ corresponding to the given feasible solution $\hat{x}$ for arc capacities u contains negative cost cycles unless $\hat{x}$ is already optimal. Hence, another interpretation of the capacity inverse problem would be to destroy the negative cycles in the residual graph $G(\hat{x}, u)$ by perturbing the arc capacities in the original graph $G = (N, A)$.

If we investigate the effects of changing the capacities of arcs in the initial graph G onto the residual graph $G(\hat{x}, u)$, we observe that there are 3 alterations that can occur in the residual graph:

1. *A new arc can be added to $G(\hat{x}, u)$:* If we increase the capacity of an arc from $u_{ij} = \hat{x}_{ij}$ to $\hat{u}_{ij} > u_{ij}$ then we create a new arc from node i to node j in $G(\hat{x}, u)$ with capacity $(\hat{u}_{ij} - \hat{x}_{ij})$.

2. *An existing arc can be deleted from $G(\hat{x}, u)$:* This takes place by decreasing the capacity $u_{ij} > \hat{x}_{ij}$ to $\hat{u}_{ij} = \hat{x}_{ij}$.

3. *The residual capacity of an already existing arc can be changed without deleting.*

Proposition 2.11. *A negative cycle in the residual graph $G(\hat{x}, u)$ can be destroyed if and only if an existing arc is deleted from $G(\hat{x}, u)$, i.e. the capacity of an arc (i, j) in the original graph G is set to its flow value $\hat{x}_{ij}$.*

Using Proposition 2.11, IMCF under ℓ_1-norm (or ℓ_∞-norm) can be reformulated as choosing a set of arcs A_D to be deleted from the residual graph $G(\hat{x}, u)$ such that $G(\hat{x}, u)$ is free of negative cycles. The objective is to find $A_D \subseteq A(\hat{x}, u)$ such that $\sum_{(i,j) \in A_D} (u_{ij} - \hat{x}_{ij})$ (or $\max_{(i,j) \in A_D} (u_{ij} - \hat{x}_{ij})$) is minimized.

While studying IMCF, the first question that arises is the feasibility of the problem. We can formulate the feasibility condition for the capacity inverse problem as follows:

Lemma 2.12. *There does not exist a feasible solution to the capacity inverse minimum cost flow problem if and only if there exists a cycle C in G with positive flows on the cycle arcs and a positive sum of arc costs, i.e.* $\forall\ (i,j) \in C \quad \hat{x}_{ij} > 0$ *and* $\sum_{(i,j)\in C} c_{ij} > 0$.

Proof: "$\Leftarrow$" The reverse cycle $\bar{C}$ of C is contained in the residual graph $G(\hat{x}, u)$ and is a negative cycle with capacities equal to the flow values $\hat{x}_{ij}$. By Proposition 2.11, any feasible solution $\hat{u}$ of the inverse problem has to break the negative cycles in $G(\hat{x}, u)$. However, independent of the choice of $\hat{u}$ the reverse cycle $\bar{C}$ remains in the residual graph since $\hat{u}_{ij} \geq \hat{x}_{ij}$.
"$\Rightarrow$" If there does not exist a feasible solution for the inverse problem (i.e., there does not exist $\hat{u}$ such that $\hat{x}$ is optimal for the min cost problem), then for all capacities $\hat{u}$ (such that $\hat{x}$ is feasible) the residual graph $G(\hat{x}, \hat{u})$ of $\hat{x}$ with respect to arc capacities $\hat{u}$ contains a negative cycle $\bar{C}$. That means deleting any residual arc on $\bar{C}$ would make the given solution $\hat{x}$ infeasible. Hence, $\bar{C}$ does not contain any forward arc $(i,j) \in A$ (otherwise $\hat{u}_{ij} = \hat{x}_{ij}$ would destroy the cycle $\bar{C}$ while preserving the feasibility of $\hat{x}$). Consequently, the reverse cycle C has $\sum_{(i,j)\in C} c_{ij} > 0$ and $\hat{x}_{ij} > 0$ holds for all arcs $(i,j) \in C$.

■

2.2.2 Rectilinear (ℓ_1) Norm

In this section, we study the complexity of the capacity inverse minimum cost flow problem under unit weight ℓ_1-norm and illustrate that the problem is $\mathcal{NP}$-complete. For this purpose, we show that the *feedback arc set problem*, which is known to be $\mathcal{NP}$-complete (Garey and Johnson, 1979; Karp, 1972), is reducible to the rectilinear IMCF (R-IMCF).

A *feedback set* in a directed graph is a set of arcs that includes at least one arc of every directed cycle. The *feedback arc set problem (FAS)* seeks a feedback set of minimum size, so that the removal of all arcs in this set makes the resulting graph acyclic. In the weighted version of FAS, there exists a nonnegative weight function associated with the arcs of the digraph. Then, the objective is to find a minimum weight set of arcs A^0 such that the directed graph $G^0 = (N, A \backslash A^0)$ is acyclic.

First, we start with the simpler case and show the reducibility of FAS (cardinality) to R-IMCF for a given integer feasible solution $\hat{x}$ on a graph with unit capacity arcs (R-1-IMCF), i.e. $u_{ij} = 1$. Since any feasible integer flow $\hat{x}$ is a 0-1 flow and $\hat{u}_{ij} \leq u_{ij}$ for all $(i,j) \in A$ by Proposition 2.11, capacities of arcs with positive flow cannot be changed, i.e. arcs in $A_R := \{(j,i) \in A(\hat{x}, u) : \hat{x}_{ij} = 1\}$ cannot be deleted from the residual graph $G(\hat{x}, u)$. The decision problems of the two problems are therefore due to Proposition 2.11 as follows:

- *FAS:* Given a directed graph $G = (N, A)$, does there exist $\tilde{A} \subseteq A$ such that $|\tilde{A}| \leq k$ and $G \backslash \tilde{A}$ is acyclic?

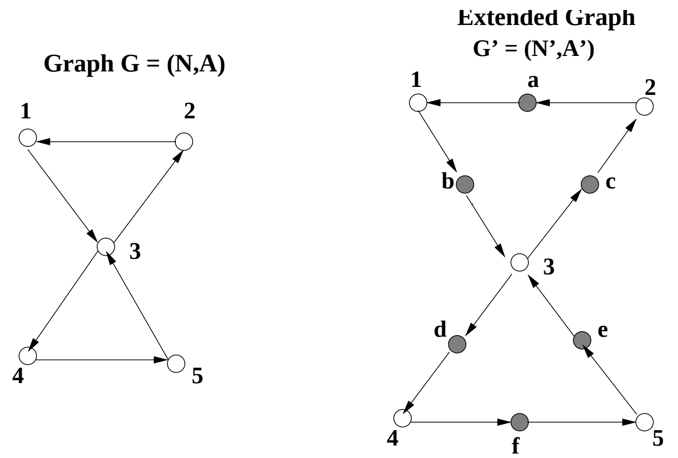

Figure 2.1: After the 1^{st} and 2^{nd} steps of transformation are applied

- *R-1-IMCF:* Given a directed graph $G' = (N', A')$ with unit arc capacities, costs associated with arcs and a subset of arcs $A_R \subsetneq A'$, does there exist $\tilde{A} \subseteq A' \backslash A_R$ such that $|\tilde{A}| \leq k$ and $G' \backslash \tilde{A}$ defines a graph that does not contain any negative cost cycles?

Given a directed graph $G = (N, A)$ with capacities $u_{ij} = 1$ for all $(i, j) \in A$, we modify graph G to obtain $G' = (N', A')$ such that G' is the residual graph for a feasible solution of a minimum cost flow problem and both graphs have the same cycles. For this purpose, we apply the following algorithm.

Algorithm 4. (Transformation Algorithm)

1. Split each arc of G into 2 arcs using a dummy node k and preserving the direction of the original arc, i.e. replace any arc (i, j) with two arcs (i, k) and (k, j) (see Figure 2.1).

2. For each arc pair $[(i, k), (k, j)]$ with dummy node k, set the cost of the arc (k, j) to $-\epsilon$ where $\epsilon > 0$. All the remaining arcs of G' will have costs of 0.

3. Add a source node s and a sink node t.

4. For each arc pair $[(i, k), (k, j)]$ where $(i, j) \in A$, add the arc from the sink node to the dummy node k of the arc pair, and the arc from the end node j to the source node. If there exist parallel arcs between node j and source node s, we add new dummy nodes d_i and replace (j, s) with the two arcs (j, d_i) and (d_i, s) (see Figure 2.2).

5. For all arcs the capacities are equal to 1.

By construction of the graph G' the following lemmata hold.

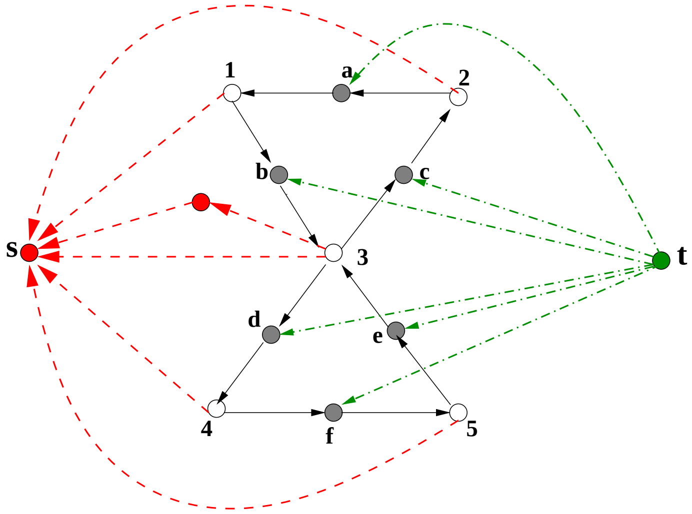

Figure 2.2: After the 3^{rd} and 4^{th} steps of transformation are applied

Lemma 2.13. *For any directed cycle of graph G, there exists a negative cost cycle in the transformed graph G' and vice versa.*

Lemma 2.14. *The directed graph G' that is constructed from the digraph G by applying the transformation algorithm defines a residual graph for a feasible solution of the minimum cost flow problem in a unit capacity graph.*

Proof: Reverse the direction of the arcs from the sink to the dummy nodes, from nodes j to the source node, and of the outgoing arcs from the dummies with negative costs (Figure 2.3). This part of the network allows a flow of value m=$|A|$ thus establishing a feasible solution to the minimum cost flow problem with a total cost of $m\epsilon$.

■

As a result of the given two lemmata (2.13 and 2.14), we can conclude that the transformation algorithm generates an instance of R-IMCF on a unit capacity graph from any given cardinality FAS. Moreover, the transformation can be done in polynomial time since the number of required changes at each step of the algorithm is bounded by $\mathcal{O}$(m). Thus, we can now show the $\mathcal{NP}$-completeness of our problem.

Theorem 2.15. *The capacity inverse minimum cost flow problem under unit weight rectilinear norm is $\mathcal{NP}$-complete on a unit capacity graph.*

Proof: Obviously, the inverse problem on graph $G' = (N', A')$ is in $\mathcal{NP}$: Given a certificate $\tilde{A} \subseteq A' \backslash A_R$, it is possible to check in polynomial time whether $G' \backslash \tilde{A}$ is free of negative cycles by using any negative cycle detecting algorithm (Ahuja *et al.*, 1993).

The transformation algorithm converts any FAS into an instance of R-IMCF in

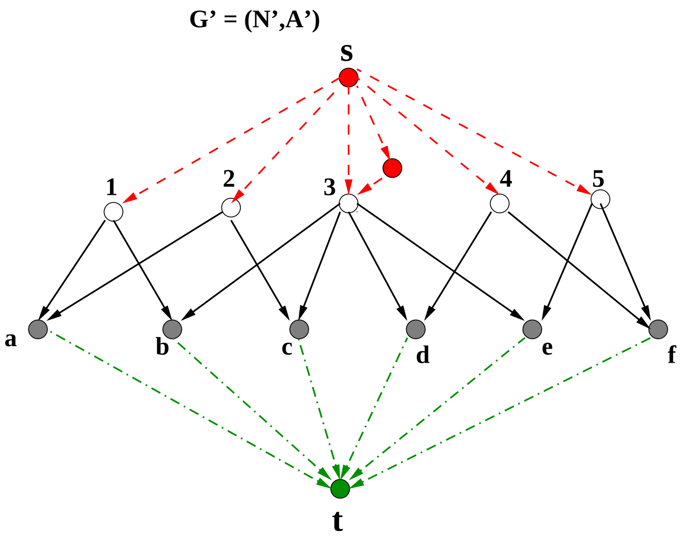

Figure 2.3: After the directions of the necessary arcs are reversed we obtain a minimum cost flow graph.

polynomial time. Hence, it is only left to prove

$$\begin{array}{rccl}
\exists \tilde{A} \subseteq A: \quad |\tilde{A}| \leq k & \text{and} & G\backslash\tilde{A} & \text{is acyclic} \\
& \Longleftrightarrow & & \\
\exists \hat{A} \subseteq A'\backslash A_R : |\hat{A}| \leq k & \text{and} & G'\backslash\hat{A} & \text{is free of negative cost cycles}
\end{array}$$

" $\Rightarrow$ " Suppose (i, j) is an arc in $\tilde{A}$. Then, by the construction of G' there exists an arc pair $[(i, d), (d, j)]$ that is the splitted version of this arc. By Lemma 2.14 one of these arcs, (d, j) with a cost of $-\epsilon$ is an element of A_R, but the other one, (i, d) can be deleted since it is not contained in A_R. So, we set $\hat{A} = \{(i, d) : (i, j) \in \tilde{A} \text{ and } (d, j) \in A_R\}$. Clearly, $|\hat{A}| = |\tilde{A}| \leq k$ and since each directed cycle of graph G is by Lemma 2.13 associated with a negative cycle in G', $G'\backslash\hat{A}$ is free of negative cycles.
" $\Leftarrow$ " Analogously we set $\tilde{A} = \{(i, j) : (i, d) \in \hat{A} \text{ and } (d, j) \in A_R\}$.

■

We can generalize this result to the rectilinear capacity inverse minimum cost flow problems for graphs with arc capacities $u_{ij} \geq 0$. The following corollary states this generalization.

Corollary 2.16. *The capacity inverse minimum cost flow problem under unit weight rectilinear norm is $\mathcal{NP}$-complete on graphs with arc capacities $u_{ij} \geq 0$.*

Proof: By using the same transformation algorithm as in the preceding case of unit capacities, we can reduce the **weighted** feedback arc set problem to R-IMCF. In this case, the weights of arcs in FAS define the capacities of the corresponding arc pairs in R-IMCF. Additionally, the capacities of the arcs $(i, s) : \; i \in N$ and $(t, j) : \; j \in N' \backslash N$ are set to be equal to the capacities of the arcs $(j, i) \in A'$. Thus, in the associated minimum cost flow problem, the flow to be sent from source node to sink node is $\sum_{(i,j)\in A} w_{ij}$, i.e. the total weight of the arcs in FAS.

■

With this corollary and the fundamental assumption $\mathcal{P} \neq \mathcal{NP}$, we can assume that R-IMCF cannot be solved in polynomial time. Hence, it is essential to find *good* approximations of the optimum solution, which gives rise to the question how well the problem R-IMCF can be approximated.

$\mathcal{APX}$ is the class of all $\mathcal{NP}$ optimization problems such that, for some $r \geq 1$, there exists a polynomial-time r-approximate algorithm for a problem P. Problem P is said to be **$\mathcal{APX}$-hard** if there exists a reduction from all problems $Q \in \mathcal{APX}$ to the problem P. If also $P \in \mathcal{APX}$, then P is called **$\mathcal{APX}$-complete**. Kann (1992) reports that the minimum weighted feedback arc set problem is $\mathcal{APX}$-hard for directed graphs and no constant approximation algorithm is known for it (Ausiello *et al.*, 1999).

In order to show the $\mathcal{APX}$-hardness of a problem, we need an **approximation preserving reduction** (Ausiello *et al.*, 1999). Let P_1 and P_2 be two optimization problems in $\mathcal{NP}$. P_1 is said to be *AP-reducible* to P_2 if two functions f and g and a positive constant $\alpha \geq 1$ exist satisfying the following conditions.

1. For any instance $x \in I_{P_1}$ and for any rational $r > 1$, $f(x, r) \in I_{P_2}$.

2. For any $x \in I_{P_1}$ and for any rational $r > 1$, if $SOL_{P_1}(x) \neq \emptyset$ then $SOL_{P_2}(f(x, r)) \neq \emptyset$.

3. For any $x \in I_{P_1}$, for any rational $r > 1$, and for any $y \in SOL_{P_2}(f(x, r))$, $g(x, y, r) \in SOL_{P_1}(x)$.

4. f and g are computable by two algorithms whose running time is polynomial for any fixed rational r.

5. For any instance $x \in I_{P_1}$, for any rational $r > 1$, and for any $y \in SOL_{P_2}(f(x, r))$, $R_{P_2}(f(x, r), y) \leq r$ implies $R_{P_1}(x, g(x, y, r)) \leq 1 + \alpha(r - 1)$.

Here, I_P is the set of instances of P, $SOL_P(x)$ is the set of feasible solutions of instance x of problem P, and $R_P(x, y)$ is the performance ratio of y with respect to x for problem P. The triple (f, g, α) is said to be an **AP-reduction** from P_1 to P_2.

Lemma 2.17. *The transformation algorithm that reduces feedback arc set problem to the rectilinear capacity inverse minimum cost flow problem is an AP-reduction in which f and g do not depend on performance ratio r and $\alpha = 1$.*

Proof: The transformation algorithm preserves the cycles and the weights of the arcs. Therefore, the feasible solutions of the two problem instances correspond to each other one-to-one. Moreover, the algorithm runs in polynomial time and the corresponding feasible solutions can be generated polynomially. The objective function value is also preserved during the transformation. Hence, it fulfills all the conditions of being an AP-reduction.

■

Since the feedback arc set problem is $\mathcal{APX}$-hard and AP-reducible to the capacity inverse minimum cost flow problem under unit-weight rectilinear norm, approximating R-IMCF is at least as difficult as approximating FAS. Thus, we can conclude the following result.

Corollary 2.18. *Capacity inverse minimum cost flow problem is $\mathcal{APX}$-hard under rectilinear norm.*

2.2.3 Chebyshev (ℓ_∞) Norm

In this section, we show that the inverse problem under ℓ_∞-norm (subsequently abbreviated with C-IMCF) is polynomially solvable using a simple greedy algorithm. At each iteration of the algorithm we select a negative cost cycle from the residual graph $G(\hat{x}, u)$ and remove an arc from this cycle. For deletion we choose an arc in the cycle which is not forbidden to be deleted and which has a minimum capacity. Note that the output of the algorithm is the new residual capacities u^*_{ij} of the arcs for which $\hat{x}$ is an optimum solution. It is a trivial computation to determine the new capacities $\hat{u}_{ij}$ of the original arcs $(i,j) \in A$ from the residual capacities u^*_{ij}.

Algorithm 5. (Greedy Algorithm)

1. Initialize the set of deleted arcs $A_D = \emptyset$ for $A_D \subseteq A(\hat{x}, u)$, where $G(\hat{x}, u) = (N, A(\hat{x}, u))$ is the input residual graph. Let the arc capacities in $G(\hat{x}, u)$ be $\bar{u}_{ij}$, $\forall (i,j) \in A(\hat{x}, u)) - A_R$ with $\bar{u}_{ij} = u_{ij} - \hat{x}_{ij}$.

2. Choose a negative cost cycle C using any negative cycle detection algorithm.
   IF there exists no negative cycle
   STOP and Output: For $(i,j) \notin A_D$ assign $u^*_{ij} = \bar{u}_{ij}$, else set $u^*_{ij} = 0$ with objective value $\max_{(i,j)\in A_D} \bar{u}_{ij}$

3. Find $(k,l) = \arg\min_{(i,j)\in C\setminus A_R} \bar{u}_{ij}$, then set $A_D = A_D \cup \{(k,l)\}$ and $A(\hat{x}, u) = A(\hat{x}, u) - \{(k,l)\}$. GO TO Step-2.

Theorem 2.19. *The greedy algorithm solves the capacity inverse minimum cost flow problem under ℓ_∞-norm optimally in $\mathcal{O}(nm^2)$ time.*

Proof: In each iteration we have to detect a negative cycle which takes $\mathcal{O}(mn)$ time (Ahuja *et al.*, 1993). Since we delete only one arc in each iteration, the algorithm will terminate after at most m iterations. Hence, the worst case running time is $\mathcal{O}(nm^2)$.

To prove the correctness of the algorithm, suppose A^* is an optimal set of arcs to be deleted for C-IMCF and $A^* \neq A_D$. Hence,

$$\max_{(i,j)\in A^*} \bar{u}_{ij} \leq \max_{(i,j)\in A_D} \bar{u}_{ij}$$

Moreover, by construction of the greedy algorithm there exists a negative cycle C for which

$$\arg\max_{(i,j)\in A_D} \bar{u}_{ij} =: (i^*, j^*) \in C \quad \text{and} \quad \bar{u}_{i^*j^*} = \min_{(i,j)\in C\setminus A_R} \bar{u}_{ij}$$

Then,

$$\max_{(i,j)\in A^*} \bar{u}_{ij} \leq \max_{(i,j)\in A_D} \bar{u}_{ij} \leq \bar{u}_{ij} \quad \forall\, (i,j) \in C\setminus A_R$$

We also conclude from Proposition 2.11 that A^* has to contain at least one arc from each negative cycle. So, there exists an arc $(k,l) \in A^* \cap C$.

$$\bar{u}_{kl} \leq \max_{(i,j)\in A^*} \bar{u}_{ij} \leq \max_{(i,j)\in A_D} \bar{u}_{ij} \leq \bar{u}_{kl} \tag{2.13}$$

Consequently, all the inequalities in (2.13) hold with equality and the solution of the greedy algorithm is optimum.

■

Although the solution of the greedy algorithm is optimal, it may not be always good enough in practice. The weakness of using only Chebyshev norm and the greedy algorithm is that some arcs might be deleted from $G(\hat{x}, u)$ unnecessarily. In other words, the solution found by the greedy algorithm does, in general, not have the minimum number of arcs to be deleted. To overcome this drawback, we use a multicriteria approach in which we exploit a lexicographic bicriteria objective function instead of the ℓ_∞-norm as a single criterion. In this approach, we first minimize the maximum capacity of the arcs to be deleted, then we minimize the number of arcs to be deleted. Hence,

$$\text{lexmin} \begin{cases} \max_{(i,j)\in A_D} \bar{u}_{ij} \\ |A_D| \end{cases}$$

generates the minimum cardinality set A_D that optimally solves C-IMCF.

The subproblem to be solved in this lexicographical problem is a rectilinear capacity inverse minimum cost flow problem with an upper bound on the capacity of the arcs to be deleted. It is not difficult to see that the rectilinear problem without upper bound constraint is a special case of the constrained problem. We only need to set an upper bound equal to the capacity of the maximum capacity arc in the graph. Since the rectilinear problem without the constraint is $\mathcal{NP}$-hard (Section 2.2.2), so is the problem with the upper bound constraint.

In the rest of this section, we propose a 2-phase approximation algorithm for solving the bicriteria inverse problem, which is based on an idea of Demetrescu and Finocchi (2003) for the weighted FAS problem.

Algorithm 6. (Algorithm FAS) The algorithm consists of two phases. First, it looks for a simple cycle $\mathcal{C}$ and, if such a cycle exists, identifies an arc in $\mathcal{C}$ having minimum weight, say ϵ. Then, the weight of all the arcs in $\mathcal{C}$ is decreased by ϵ and the arcs whose weight becomes zero are removed. The first phase terminates when the graph becomes acyclic. In the second phase, the algorithm tries to add back some of the deleted arcs to the graph paying attention that no cycles are reintroduced. (Demetrescu and Finocchi, 2003)

The idea of the FAS algorithm can be adapted to approximate an efficient solution for the lexicographic bicriteria problem after making some modifications. Our version of the algorithm runs in the following way. In the first phase, we find an optimal solution for the Chebyshev inverse problem by applying the greedy algorithm. In this phase we also keep track of the **cycle index**, i.e. number of cycles an arc is included in. This number is clearly a lower bound on the number of cycles containing a certain arc. In Phase 2 we start with the feasible set of deleted arcs found by the greedy phase and apply two methods to reduce the number of arcs deleted while maintaining the feasibility:

1. We choose an arc (i, j) with the minimum cycle index and reinsert it to the graph if no new negative cycle is introduced.

2. If arc (i, j) cannot be reinserted to the graph, we find a negative cycle including arc (i, j). We select a new arc (k, l) on this cycle with the same or higher cycle index and which has a capacity not greater than the optimal solution of the Chebyshev problem. Then we delete the new arc (k, l) from the residual graph if it remains negative cycle free. We call this operation **SWAP**.

Below we provide a pseudo-code of the algorithm, which we call **bicriteria approximation algorithm.**

Algorithm 7. (Bicriteria Approximation Algorithm)

- 1^{st} **Phase:** Apply Greedy Algorithm to find $\bar{u}_{max}$ and a feasible arc set A_D
- 2^{nd} **Phase:**
  - **FOR ALL DELETED ARCS**
    - **INSERTION 1:**
      - Choose a deleted arc (i, j) with the minimum cycle index among the non-processed ones

- **IF** inserting (i,j) causes no cycles, set $A(\hat{x},u) = A(\hat{x},u) \cup \{(i,j)\}$ and mark (i,j) processed.
- **ELSE** go to SWAP

* **SWAP:**
  - Find a negative cycle C that contains the arc (i,j)
  - Find a new arc $(k,l) \in C$ such that cycle index$(k,l) \geq$ cycle index(i,j), $A_D \backslash \{(i,j)\} \cup \{(k,l)\}$ is feasible, and $\bar{u}_{kl} \leq \bar{u}_{max}$
  - **IF** (k,l) exists, set $A_D = A_D \backslash \{(i,j)\} \cup \{(k,l)\}$ and mark (k,l) processed.
  - **ELSE** go to INSERTION 1

– **FOR ALL DELETED ARCS**

* **INSERTION 2:**
  - Choose a deleted arc (i,j) with the minimum cycle index
  - **IF** inserting (i,j) does not cause cycles, set $A(\hat{x},u) = A(\hat{x},u) \cup \{(i,j)\}$

Obviously, the algorithm has a polynomial running time. It has been already shown that the first phase has a worst case running time of $\mathcal{O}(nm^2)$. In the second phase the most time consuming operation is SWAP. Testing the feasibility of a new arc in SWAP requires $\mathcal{O}(nm)$ time. This test is applied in the worst case for m arcs and SWAP runs at most m times. So the worst case running time of the second phase is $\mathcal{O}(nm^3)$.

2.2.4 Computational Experiments

In the empirical analysis of our heuristic, a crucial part is the generation of suitable test cases. Since the bicriteria problem is $\mathcal{NP}$-hard, it is desirable to have input residual graphs for which the optimal number of arcs to be deleted is known. For this purpose we exploit the method described by Saab (2001) for feedback arc set problems after making necessary changes.

Avg. Degree	# Nodes	Optimal # Deleted Arcs
4	500	90
	1000	150
	2000	210
8	500	120
	1000	200
	1500	250
16	500	180
	1000	250

Table 2.1: Graphs generated by the modification of Saab's method (Saab, 2001)

The original algorithm plants arc disjoint cycles by using the relationship between vertex ordering and feedback arc sets. Graphs including only arc disjoint cycles are certainly inappropriate for testing the bicriteria algorithm, because the set of deleted

# Nodes	# Arcs	Index	Optimal	Bicriteria Approx.
96	528	1	21	21
		2	26	28
		3	31	33
		4	33	33
		5	44	46
160	912	1	14	14
		2	18	18
		3	29	29
		4	31	32
		5	46	49
200	1340	1	22	22
		2	29	31
		3	33	35
		4	42	43
		5	47	56
		6	61	65
500	3975	1	8	8
		2	20	21
		3	20	22
		4	31	34
		5	40	44

Table 2.2: Optimal test cases generated from RMFGEN instances (Goldfarb and Grigoriadis, 1988)

arcs generated in the greedy phase is optimum. In the new version of the algorithm we plant arc disjoint cycle groups such that the optimum number of deleted arcs is equal to the number of these cycle groups. Within these cycle groups, all the cycles have exactly one arc in common (leftward arc in vertex ordering). Moreover, we allow subgroups inside these cycle groups. We force one of the leftward arcs to have the highest capacity amongst the leftward arcs but to be the minimum capacity arc of its cycle group. The rest of the capacities are randomly generated such that the desired optimality is preserved. Random negative costs are assigned to the arcs.

We implemented the bicriteria approximation algorithm using C++ and the Boost Graph Library (Siek *et al.*, 2002), and tested on graphs constructed by the modified version of Saab's algorithm. In Table 2.1 we provide the average node degree, number of nodes and the optimal number of deleted arcs for these graphs. We generated for each size 3 instances, so 24 graphs in total. We observed that all these test cases are solved to optimality by the bicriteria approximation algorithm. The reason for this lies in the structure of the graphs generated. In these graphs all the cycle groups constructed contain at least one negative cost 2-cycle or 3-cycle. This indicates that as the number of arc disjoint cycles increases and the length of negative cycles decreases the algorithm performs better independent of the size and degree of the graph.

Next, we tested the bicriteria algorithm for different types of graphs in which the negative cost cycles are irregularly distributed. For this purpose we generated minimum cost flow problems and corresponding feasible, but non-optimal solutions. The major difficulty here was to identify the optimal solution of each test case. In order to overcome this difficulty we modeled the problem as a set covering problem. We used the set of negative cost cycles identified in the greedy phase to initialize the constraint set of the set covering problem. Then, we calculated the optimal solution by iteratively solving the set covering problem and adding each time a new negative cost cycle to the constraint set if the graph was not negative cycle free. Following Caprara *et al.* (2000), we implemented the proposed set covering scheme using CPLEX 9.2 and C++.

# Nodes	# Arcs	Index	Optimal	Bicriteria Approx.
200	1308	1	38	43
	1500	2	49	60
	2000	3	19	26
	2200	4	39	46
	2900	5	30	31
300	3174	1	42	45
	4519	2	46	51
	5168	3	49	51
	6075	4	37	44
350	4508	1	56	61
	6000	2	53	56
	9000	3	58	60

Table 2.3: Optimal test cases generated from NETGEN instances (Klingman *et al.*, 1974)

The minimum cost flow problems were generated on 2 different types of graphs. The first group contains grid structured RMFGEN graphs (Goldfarb and Grigoriadis, 1988). These graphs are sparse with a single source and a single sink and they may contain 2-cycles. The second group consists of the well-known NETGEN networks. These graphs are transportation networks in which half of the nodes are sources and the other half are sinks having varying average node degrees.

Table 2.2 and Table 2.3 illustrate the approximation results for RMFGEN and NETGEN graphs, respectively. For 33% of the RMFGEN test cases the bicriteria algorithm could compute the optimal solution. Moreover, the highest relative error is less than 20%. On the other hand, for NETGEN test cases the bicriteria algorithm could not find any optimal solutions and the highest relative error is almost 40%. Although the number of test cases is limited, one can observe that the performance of the bicriteria algorithm depends on the structure of the graph. As expected, for graphs containing shorter negative cost cycles, the algorithm computes better approximations.

Apparently, the set covering scheme cannot always compute an optimal solution in a reasonable time, which we accepted as 10 minutes. For these cases, we modified the set covering scheme so that we can find a feasible solution that serves as an approximation. The pseudo-code of the set covering approximation is as follows:

Algorithm 8. (Set Covering Approximation)

1. Apply the Greedy Algorithm to find $\bar{u}_{max}$ and a set of negative cost cycles $\mathcal{C}$
2. Initialize a set covering problem S_1 for the residual graph $G(\hat{x}, u) = (N, A(\hat{x}, u))$ with the cycle set $\mathcal{C}_1 = \mathcal{C}$
3. **Iteration i:**
   - Solve set covering problem S_i and find a set of deleted arcs D_i
   - **If** $G_i(\hat{x}, u) = (N, A(\hat{x}, u)\backslash D_i)$ is negative cycle free, STOP: output D_i
   - **Else if** Time < 5 min. and C_{new} is a negative cycle in $G_i(\hat{x}, u) = (N, A(\hat{x}, u)\backslash D_i)$ **then** set $\mathcal{C}_{i+1} = \mathcal{C}_i \cup C_{new}$. Go to iteration $i + 1$.
   - **Else** set $G(\hat{x}, u) = (N, A(\hat{x}, u)\backslash D_i)$, initialize $\mathcal{C}$, go to step 2.

# Nodes	# Arcs	Index	Set Covering Approx.	Bicriteria Approx.
200	1340	1	141	141
		2	80	86
		3	126	126
		4	233	229
		5	149	170
500	3975	1	155	141
		2	101	101
		3	40	39
		4	78	77
		5	55	55

Table 2.4: Approximated test cases generated from RMFGEN instances (Goldfarb and Grigoriadis, 1988)

# Nodes	# Arcs	Index	Set Covering Approx.	Bicriteria Approx.
200	1308	1	134	130
	1500	2	82	97
	2000	3	107	113
	2200	4	281	291
	2900	5	87	96

Table 2.5: Approximated test cases generated from NETGEN instances (Klingman *et al.*, 1974)

Table 2.4 and 2.5 compare the approximation values of two algorithms for different graphs. In most of the cases, bicriteria approximation performs as well as the set covering approximation. Moreover, the bicriteria approximation has an obvious time advantage over the set covering approximation. The shortest running time for set covering approximation is 5 minutes whereas this is the highest running time for the bicriteria approximation scheme.

Looking at the computational test results we can conclude that it is reasonable to use the bicriteria approximation algorithm when the running time is critical or the graph has a simple structure with several 2 and 3-cycles. However, for small and moderate size graphs with complex structure we suggest to employ a set covering formulation implemented with CPLEX if it is essential to find an exact solution and there are no constraints on CPU time.

2.2.5 Extension to Flows with Lower and Upper Bounds

In this section we consider the minimum cost flow problem with upper u_{ij} and lower l_{ij} bounds on the flows with $l_{ij} \leq u_{ij}$ for all $(i,j) \in A$ where l_{ij} is not necessarily equal to 0. In this version of capacity inverse problem, we are allowed to perturb both the upper and lower bounds of the arcs such that the given feasible flow $\hat{x}$ will be optimum with respect to the new upper and lower bounds $(\hat{u}, \hat{l})$.

Lemma 2.20. *The capacity inverse minimum cost flow problem with upper and lower bounds has always a feasible solution.*

Proof: If we assign $\hat{u}_{ij} = \hat{l}_{ij} = \hat{x}_{ij}$ for all $(i,j) \in A$, we find a feasible solution to the capacity inverse minimum cost flow problem.

Corollary 2.21. *The capacity inverse minimum cost flow problem with lower and upper bounds is*

- *$\mathcal{NP}$-complete under ℓ_1-norm,*
- *polynomially solvable with the greedy algorithm of Section 2.2.3 under ℓ_∞-norm.*

Proof: If we have upper and lower bounds for the flows, then the alterations that can occur in the residual graph by changing a lower or an upper bound is as follows:

1. *A new arc can be added to $G(\hat{x}, u)$:* If we increase the upper capacity of an arc from $u_{ij} = \hat{x}_{ij}$ to $\hat{u}_{ij} > u_{ij}$ or decrease the lower capacity from $l_{ij} = \hat{x}_{ij}$ to $\hat{l}_{ij} < l_{ij}$, then we create a new arc from node i to node j or from j to i in $G(\hat{x}, u)$ with capacity $(\hat{u}_{ij} - \hat{x}_{ij})$ or $(\hat{x}_{ij} - \hat{l}_{ij})$.
2. *An existing arc can be deleted from $G(\hat{x}, u)$:* This takes place by decreasing the upper capacity $u_{ij} > \hat{x}_{ij}$ to $\hat{u}_{ij} = \hat{x}_{ij}$ or increasing the lower capacity $l_{ij} < \hat{x}_{ij}$ to $\hat{l}_{ij} = \hat{x}_{ij}$.
3. *The residual capacity of an already existing arc can be changed without deleting.*

Since we would like to get rid of the negative cost cycles from $G(\hat{x}, u)$, we need to delete arcs, which can be achieved by applying the 2^{nd} alteration. Hence, all the previous results of the capacity inverse minimum cost flow problems with $l_{ij} = 0$ is valid for the capacity inverse minimum cost flow problems with $l_{ij} \neq 0$ except that in the latter problem there does not exist any restricted set of arcs in $G(\hat{x}, u)$, i.e., $A_R = \emptyset$.

■

When a traveler reaches a fork in the road, the ℓ_1-norm tells him to take either one way or the other, but the ℓ_2-norm instructs him to head off into the bushes.

Claerbout & Muir (1973)

3 Inverse Tension Problems

Within the area of inverse optimization inverse network flows have been intensely investigated. In contrast, *tension problems*, which are duals of flow problems (Ahuja *et al.*, 1993), and their inverse versions have vastly been neglected. Our aim in this chapter is to fill this gap in the literature and extend the results of Ahuja and Orlin (2002) for tensions to show that the duality relation between tensions and flows is valid for their respective inverse problems, as well. Furthermore, studying inverse tension problems on networks provides the means to achieve a well-established generalization of inverse network problems to matroid flows and monotropic programs, which will be done in Chapters 4 and 5.

First, we briefly explain some basics of tension problems. For more details we refer to the textbook of Rockafellar (1984). Let $G = (N, A)$ be a connected digraph with a node set N containing n nodes and an arc set A containing m arcs, and (i, j) represent an arc with tail node i and head node j. A *tension* is a function from A to $\mathbb{R}^{|A|}$ which satisfies Kirchhoff's law for voltages (Pla, 1971). In other words, a vector $\theta \in \mathbb{R}^{|A|}$ is a *tension* on graph G with *potential* $\pi \in \mathbb{R}^{|N|}$ such that $\theta_{ij} = \pi(j) - \pi(i)$ for all $(i, j) \in A$. The basic properties of the tensions are as follows:

- For all cycles C, $\sum_{(i,j)\in C^+} \theta_{ij} - \sum_{(i,j)\in C^-} \theta_{ij} = 0$, where C^+ and C^- are the forward and the backward arcs of the cycle, respectively.

- Any linear combination of tensions is a tension.

- A tension is orthogonal to any circulation.

Minimum cost tension problem (MCT) is finding a tension θ satisfying lower ($t_{ij} \in \mathbb{R} \cup \{-\infty\}$) and upper ($T_{ij} \in \mathbb{R} \cup \{+\infty\}$) bounds on each arc such that the total cost $\sum_{(i,j)\in A} c_{ij}\theta_{ij}$ is minimum. In *maximum tension problem (MaxT)*, graph G contains 2 special nodes, s and t, and an arc $(s,t) \in A$ between these two nodes with bounds $(t_{st}, T_{st}) = (-\infty, \infty)$. The maximum tension problem is finding the maximum tension on arc $(s,t) \in A$ such that the tensions on all arcs satisfy the upper and lower bounds. In this study we assume that both problems are feasible and have finite optimal solutions. Our aim is to analyze their inverse versions.

Given a feasible tension $\hat{\theta}$ for an instance of MCT, the *cost inverse minimum cost tension problem* (IMCT$_c$) is to perturb the cost vector from c to $\hat{c}$ in a way that $\hat{\theta}$ will become an optimum tension for the minimum cost tension problem with the perturbed cost vector (MCT($\hat{c}$)) while the perturbation $\|c - \hat{c}\|$ is minimized according to some norm.

On the other hand, in the *capacity inverse minimum cost tension problem* (IMCT$_t$) we modify the bound vectors from T to $\hat{T}$ and from t to $\hat{t}$ such that $\hat{\theta}$ will become an optimum tension for the minimum cost tension problem with the perturbed bound vectors (MCT($\hat{T}, \hat{t}$)) while the perturbation is minimized according to some norm. Similarly, in *inverse maximum tension problem* (IMaxT) the bound vectors are perturbed from T to $\hat{T}$ and from t to $\hat{t}$ such that $\hat{\theta}_{st}$ will become a maximum tension with the perturbed bound vectors.

We exploit rectilinear (ℓ_1) and Chebyshev (ℓ_∞) norms to measure the parameter modifications for IMaxT and IMCT$_c$, whereas we consider only the Chebyshev norm for IMCT$_t$ since the flow version of this problem is already $\mathcal{NP}$-hard under rectilinear norm. Note that throughout this chapter $\hat{\theta}$, c and (t, T) are part of the data while $(\hat{t}, \hat{T})$ and $\hat{c}$ are the vectors of variables to be determined.

The rest of this chapter is organized as follows: In Section 3.1 we analyze the inverse maximum tension problem under rectilinear and Chebyshev norms. Section 3.2 describes the cost inverse minimum cost tension problem under ℓ_1 and ℓ_∞ norms in detail and gives a combinatorial formulation of the problems. The capacity inverse minimum cost tension problem, in which we only perturb the bounds, is investigated in Section 3.3.

3.1 Inverse Maximum Tension Problem

3.1.1 Rectilinear (ℓ_1) Norm

Yang *et al.* (1997) study inverse maximum flow problem and show that for unit weight case this problem can be reduced to solving a maximum flow problem. More recently, Deaconu (2008) have extended the results of Yang *et al.* (1997) for the maximum flow problems with upper and lower bounds for the flow where both bounds are allowed to be changed within a certain interval. In this part we will show a similar result for inverse maximum tension problem under ℓ_1-norm but we do not restrict the increase or decrease of the bounds.

Given a positive weight vector w for changing the bounds of the arcs, the inverse maximum tension problem under ℓ_1-norm is

$$\min \sum_{(i,j)\in A} w_{ij}(|\hat{T}_{ij} - T_{ij}| + |\hat{t}_{ij} - t_{ij}|) \tag{3.1}$$

subject to

$$\hat{t}_{ij} \leq \hat{\theta}_{ij} \leq \hat{T}_{ij} \quad \forall\, (i,j) \in A,$$
$$\hat{\theta}_{st} \quad \text{is the maximum tension.}$$

The maximum tension problem is the dual of the maximum flow problem, and so is the optimality condition (Rockafellar, 1984).

Theorem 3.1. ***(Max-Tension Min-Path Theorem)*** *Suppose there is at least one tension satisfying the upper and lower bounds. Then, the maximum in a maximum tension problem is equal to the minimum in the corresponding minimum path problem. Both of the problems are unbounded if there is an $s-t$ cut ω with an unlimited span, i.e. all forward arcs have infinite upper bounds and all backward arcs have infinite lower bounds.*

By Theorem 3.1 we know that there exists a minimum path, which has a length equal to the maximum tension. Moreover, for this minimum path the following property holds.

Property 3.2. *If P denotes the minimum path between s and t on graph G and P^+ and P^- are the corresponding sets of forward and backward arcs on P, then $\theta^*_{ij} = T_{ij}$ for all $(i,j) \in P^+$ and $\theta^*_{ij} = t_{ij}$ for all $(i,j) \in P^-$ for a maximum tension θ^*.*

Lemma 3.3. *If Problem (3.1) has an optimal solution (t^*, T^*) and P^* is the minimum $s-t$ path in network $G = (N, A, t^*, T^*)$, then*

1. $t \leq t^* \leq T^* \leq T$.
2. $T^*_{ij} = T_{ij}$ *and* $t^*_{ij} = t_{ij}$ *for each arc* $(i,j) \notin P^*$. *Moreover,* $t^*_{ij} = t_{ij}$ *for all arcs* $(i,j) \in P^{*+}$ *and* $T^*_{ij} = T_{ij}$ *for all arcs* $(i,j) \in P^{*-}$.

Proof:

1. Since $\hat{\theta}$ is a maximum tension in $G(t^*, T^*)$, $\hat{\theta}_{ij} = T^*_{ij}$ for all $(i,j) \in P^{*+}$ and $\hat{\theta}_{ij} = t^*_{ij}$ for all $(i,j) \in P^{*-}$ by Property 3.2. Moreover, $t^* \leq T^*$ because of feasibility.

   If there is an arc $(k,\ell) \in A$ with $T^*_{k\ell} > T_{k\ell}$ (or $t^*_{k\ell} < t_{k\ell}$), then obviously $(k,\ell) \notin P^*$ since otherwise $\hat{\theta}$ cannot be a feasible tension in $G(t,T)$. We define the new bound vectors as follows:

$$\bar{T}_{ij} = \begin{cases} T^*_{ij} & \text{if } (i,j) \neq (k,\ell) \\ T_{ij} & \text{if } (i,j) = (k,\ell) \end{cases} \qquad \bar{t}_{ij} = \begin{cases} t^*_{ij} & \text{if } (i,j) \neq (k,\ell) \\ t_{ij} & \text{if } (i,j) = (k,\ell) \end{cases}$$

By Property 3.2, it is easy to verify that $\hat{\theta}$ is a maximum tension under $(\bar{t}, \bar{T})$. Moreover,

$$\sum_{(i,j)\in A} w_{ij}(|\bar{T}_{ij} - T_{ij}| + |\bar{t}_{ij} - t_{ij}|) < \sum_{(i,j)\in A} w_{ij}(|T^*_{ij} - T_{ij}| + |t^*_{ij} - t_{ij}|)$$

which is a contradiction to the optimality of (t^*, T^*). Hence, the result follows.

2. Let us define the bound vectors $(\bar{t}, \bar{T})$ as follows:

$$\bar{T}_{ij} = \begin{cases} T^*_{ij} & \text{if } (i,j) \in P^{*+} \\ T_{ij} & \text{otherwise} \end{cases} \qquad \bar{t}_{ij} = \begin{cases} t^*_{ij} & \text{if } (i,j) \in P^{*-} \\ t_{ij} & \text{otherwise} \end{cases}$$

By Property 3.2, $\hat{\theta}$ remains a maximum tension under $(\bar{t}, \bar{T})$. Since

$$\sum_{(i,j)\in A} w_{ij}(|\bar{T}_{ij} - T_{ij}| + |\bar{t}_{ij} - t_{ij}|) \leq \sum_{(i,j)\in A} w_{ij}(|T^*_{ij} - T_{ij}| + |t^*_{ij} - t_{ij}|) \tag{3.2}$$

and (t^*, T^*) is an optimum solution of the inverse maximum tension problem, the inequality (3.2) holds with equality and the conclusion is true.

■

Recall that the given tension $\hat{\theta}$ is a feasible tension for $G(t,T)$, thus $\hat{\theta} < \theta^*$ where θ^* is an optimum tension for $G(t,T)$. By using this fact and Lemma 3.3 we can reformulate IMaxT under ℓ_1-norm as follows:

Lemma 3.4. *The inverse maximum tension problem under ℓ_1-norm is equivalent to finding a path P from s to t in $G = (N, A)$ such that*

$$\sum_{(i,j)\in P^+} w_{ij}(T_{ij} - \hat{\theta}_{ij}) + \sum_{(i,j)\in P^-} w_{ij}(\hat{\theta}_{ij} - t_{ij})$$

is minimum.

Theorem 3.5. *Let P^* is the minimum path corresponding to the maximum tension problem in $G(t,T)$. The optimum solution of the inverse maximum tension problem with respect to unit weight ℓ_1-norm is*

$$T^*_{ij} = \begin{cases} \hat{\theta}_{ij} & \text{if } (i,j) \in P^{*+} \\ T_{ij} & \text{otherwise} \end{cases} \qquad t^*_{ij} = \begin{cases} \hat{\theta}_{ij} & \text{if } (i,j) \in P^{*-} \\ t_{ij} & \text{otherwise} \end{cases}$$

Hence, solving the inverse problem is equivalent to solving a maximum tension problem on $G(t,T)$.

Proof: If arc weights $w_{ij} = 1$ for all arcs $(i,j) \in A$, then by Lemma 3.4 we need to find a path P minimizing

$$\sum_{(i,j)\in P^+} (T_{ij} - \hat{\theta}_{ij}) + \sum_{(i,j)\in P^-} (\hat{\theta}_{ij} - t_{ij}).$$

By rearranging this objective function, we obtain

$$\left(\sum_{(i,j)\in P^+} T_{ij} - \sum_{(i,j)\in P^-} t_{ij} \right) - \left(\sum_{(i,j)\in P^+} \hat{\theta}_{ij} - \sum_{(i,j)\in P^-} \hat{\theta}_{ij} \right).$$

Since the value of $\hat{\theta}$ between s and t is a given constant, the problem is equivalent to identifying a minimum path on $G(t,T)$ and the result follows.

■

Theorem 3.6. *The solution to the inverse maximum tension problem under ℓ_1-norm with a positive weight function w can be found by solving a maximum tension problem in graph G with respect to upper and lower bounds $T^*_{ij} := w_{ij}(T_{ij} - \hat{\theta}_{ij})$ and $t^*_{ij} := w_{ij}(t_{ij} - \hat{\theta}_{ij})$ on arcs $(i,j) \in A \backslash \{(s,t)\}$, respectively.*

Proof: The maximum tension problem on G with upper bounds $w_{ij}(T_{ij} - \hat{\theta}_{ij})$ and lower bounds $w_{ij}(t_{ij} - \hat{\theta}_{ij})$ for $(i,j) \in A \backslash \{(s,t)\}$ is feasible since

$$t^*_{ij} := w_{ij}(t_{ij} - \hat{\theta}_{ij}) \leq 0 \leq w_{ij}(T_{ij} - \hat{\theta}_{ij}) =: T^*_{ij}.$$

Moreover, the length of the minimum path P is

$$\sum_{(i,j)\in P^+} w_{ij}(T_{ij} - \hat{\theta}_{ij}) - \sum_{(i,j)\in P^-} w_{ij}(t_{ij} - \hat{\theta}_{ij}) \tag{3.3}$$

which is by Lemma 3.4 a solution to the inverse maximum tension problem.

■

The maximum tension problem on a graph can be solved in polynomial time by using the Minimum Path Algorithm (Rockafellar, 1984).

Algorithm 9. (Minimum Path Algorithm)

Input: An instance of a maximum tension problem on $G = (N, A)$ with bounds (t, T) and an initial feasible tension θ^0 together with the corresponding node potentials π^0.

Output: A maximum tension between s and t.

1. Set the artificial spans $T^0_{ij} := T_{ij} - \theta^0_{ij}$ and $t^0_{ij} := t_{ij} - \theta^0_{ij}$ for all $(i,j) \in A$ and initialize $S := \{s\}$ and $p(i) := 0$ for all $i \in N$.

2. IF $t \in S$, STOP. The optimum node potentials are $\pi = \pi^0 + p$.
   ELSE for the cut $\omega = (S, N \backslash S)$ calculate
$$\beta = \min \begin{cases} p(i) + T^0_{ij} & \text{for } (i,j) \in \omega^+, \\ p(i) - t^0_{ij} & \text{for } (i,j) \in \omega^-. \end{cases}$$
3. IF $\beta = \infty$, then STOP. ω is a cut of unlimited span.
4. IF $\beta < \infty$, take any arc $(i,j) \in \omega$ achieving the minimum β.
   SET $S := S \cup \{j\}$ and $p(j) := \beta$. (If there are more arcs satisfying the minimum, then several nodes can be added at once.)
   GO TO Step-2.

In his book, Rockafellar (1984) also provides a faster version of the Minimum Path Algorithm, which keeps track of the *scanned* nodes in S. This faster version is indeed a modified Dijkstra's Algorithm.

Recall that in the nonunit weights case of the inverse maximum tension problem (Theorem 3.6) the lower and upper bounds of the corresponding maximum tension problem satisfy $t^* \leq 0 \leq T^*$. Hence, $\theta^0 = 0$ is an initial feasible solution and we can directly apply the Minimum Path Algorithm. For the unit weights case (Theorem 3.5), we need to identify an initial feasible tension. This can be achieved by applying the Feasible Differential Algorithm of Rockafellar (1984), which starts with an arbitrary set of node potentials and applies a slightly modified version of the Minimum Path Algorithm iteratively until the feasibility is fulfilled. The worst case complexity of this algorithm is $\mathcal{O}(n^3)$. Here, we do not provide any more details of this algorithm and, instead, refer to the original book.

3.1.2 Chebyshev (ℓ_∞) Norm

In this section we analyze the maximum tension problem under Chebyshev norm and prove that this problem can be solved as maximum capacity path problem similar to the flow case, which was treated in Section 2.1.

IMaxT under ℓ_∞-norm can be formulated as follows

$$\begin{aligned} &\min \max_{(i,j)\in A} w_{ij}(\max\{|\hat{T}_{ij} - T_{ij}|, |\hat{t}_{ij} - t_{ij}|\}) \qquad (3.4) \\ &\text{subject to} \\ &\qquad \hat{t}_{ij} \leq \hat{\theta}_{ij} \leq \hat{T}_{ij} \quad \forall\, (i,j) \in A, \\ &\qquad \hat{\theta}_{st} \quad \text{is the maximum tension.} \end{aligned}$$

Similar to the rectilinear case, we can prove the correctness of Lemma 3.3 for the ℓ_∞-norm by exploiting Theorem 3.1 and Property 3.2. As a consequence of Lemma 3.3, we know that we need to either reduce the upper bound of an arc or increase the lower bound so that one of the $s-t$ paths will be a minimum path while the objective function of (3.4) is satisfied. This leads to the following conclusion:

Lemma 3.7. *The inverse maximum tension problem under ℓ_∞-norm is equivalent to finding a path P from s to t in $G=(N,A)$ such that*

$$\max\{\max_{(i,j)\in P^+} w_{ij}(T_{ij}-\hat{\theta}_{ij}), \max_{(i,j)\in P^-} w_{ij}(\hat{\theta}_{ij}-t_{ij})\}$$

is minimum among all $s-t$ paths.

In order to find the $s-t$ path P mentioned in Lemma 3.7, we define a new graph $\bar{G}=(N,\bar{A})$ with $\bar{A}=A\cup\{(i,j):(j,i)\in A\}$. Moreover, we assign arc capacities of

$$u_{ij} := \begin{cases} w_{ij}(\hat{\theta}_{ij}-T_{ij}) & \text{if} \quad (i,j)\in A, \\ w_{ji}(t_{ji}-\hat{\theta}_{ji}) & \text{if} \quad (i,j)\in \bar{A}\backslash A. \end{cases} \tag{3.5}$$

By the construction of the graph $\bar{G}=(N,\bar{A})$, the following proposition holds:

Proposition 3.8. *Let $\bar{\mathcal{P}}$ be the set of all directed (elementary) $s-t$ paths on $\bar{G}$ and $\mathcal{P}$ be the set of all (elementary) $s-t$ paths in G. For each path $P\in\mathcal{P}$, there exists a path $\bar{P}\in\bar{\mathcal{P}}$ such that*

$$\bar{P}=\{(i,j):(i,j)\in P^+\}\cup\{(i,j):(j,i)\in P^-\}$$

and vice versa.

Now we can prove the main conclusion of this section.

Theorem 3.9. *Let $P^*\in\bar{\mathcal{P}}$ be a maximum capacity directed (elementary) path on $\bar{G}=(N,\bar{A})$ with a capacity of u^* (i.e. u^* is the capacity of a minimum capacity arc on P^*). Then, an optimal solution (t^*,T^*) of the inverse maximum tension problem under ℓ_∞-norm is*

$$T^*_{ij} = \begin{cases} \hat{\theta}_{ij} & \textit{if } (i,j)\in P^*\cap A \\ T_{ij} & \textit{otherwise} \end{cases} \qquad t^*_{ij} = \begin{cases} \hat{\theta}_{ij} & \textit{if } (j,i)\in P^*\cap(\bar{A}\backslash A) \\ t_{ij} & \textit{otherwise} \end{cases} \tag{3.6}$$

with an objective function value of $-u^*$.

Proof: Since P^* is a maximum capacity directed path on $\bar{G}$, the capacity u^* satisfies

$$u^* = \max_{\bar{P}\in\bar{\mathcal{P}}}\min\left\{\min_{(i,j)\in\bar{P}\cap A} w_{ij}(\hat{\theta}_{ij}-T_{ij}), \min_{(i,j)\in\bar{P}\cap(\bar{A}\backslash A)} w_{ji}(t_{ji}-\hat{\theta}_{ji})\right\}$$

If we multiply both sides with -1, the maximum and minimum change places and we obtain

$$-u^* = \min_{\bar{P}\in\bar{\mathcal{P}}}\max\left\{\max_{(i,j)\in\bar{P}\cap A} w_{ij}(T_{ij}-\hat{\theta}_{ij}), \max_{(i,j)\in\bar{P}\cap\bar{A}\backslash A} w_{ji}(\hat{\theta}_{ji}-t_{ji})\right\}.$$

By Proposition 3.8, there exists an elementary $s-t$ path $P \in \mathcal{P}$ such that $P^+ = \bar{P} \cap A$ and $P^- = \{(i,j) \in A : (j,i) \in \bar{P} \cap (\bar{A} \backslash A)\}$. Hence,

$$-u^* = \min_{P \in \mathcal{P}} \max \left\{ \max_{(i,j) \in P^+} w_{ij}(T_{ij} - \hat{\theta}_{ij}), \max_{(i,j) \in P^-} w_{ij}(\hat{\theta}_{ij} - t_{ij}) \right\}.$$

By Lemma 3.7, $-u^*$ is the objective function value of the inverse maximum tension problem under ℓ_∞-norm and (t^*, T^*) is an optimal solution by Theorem 3.1 and Property 3.2.

■

Note that the arc capacities in graph $\bar{G}$ are nonpositive real numbers, but we can scale the capacities in order to apply the algorithms solving maximum capacity path problem (i.e. add $|U|+1$ to the capacities where U is the most negative capacity on the graph $\bar{G}$).

3.2 Cost Inverse Minimum Cost Tension Problem

3.2.1 Rectilinear (ℓ_1) Norm

For the inverse minimum cost flow problem under unit weight ℓ_1-norm, Ahuja and Orlin (2002) showed that the optimum objective function value is equal to the minimum cost of a collection of arc-disjoint cycles on the corresponding residual graph. Since this collection defines a minimum cost circulation in a unit capacity network, the inverse problem can be reduced to solving a minimum cost flow problem in a unit capacity network. Similarly, by using *arc-disjoint* residual cuts, we will show that the inverse minimum cost tension problem under unit weight rectilinear norm reduces to solving a minimum cost tension problem with unit upper and lower bounds on arcs.

In this section, we are given a minimum cost tension problem (MCT) on a directed graph $G = (N, A)$ with upper and lower bound vectors $(T, t) \in (\mathbb{R}^{|A|}, \mathbb{R}^{|A|})$ and a cost vector $c \in \mathbb{R}^{|A|}$. Let $\hat{\theta}$ be a nonoptimal feasible solution to this MCT. The cost inverse minimum cost tension problem under unit weight ℓ_1-norm is

$$\min \sum_{(i,j) \in A} |\hat{c}_{ij} - c_{ij}| \tag{3.7}$$

subject to

$$t_{ij} \leq \hat{\theta}_{ij} \leq T_{ij} \quad \forall\, (i,j) \in A,$$

$\hat{\theta}$ is a minimum cost tension with respect to $\hat{c}$.

In order to replace the verbal formulation in (3.7) with a mathematical one, we need to first define arc-disjoint residual cuts and present the optimality conditions for minimum cost tensions.

A cut ω is called ***residual*** with respect to a tension $\hat{\theta}$ if

$$\forall\, (i,j) \in \omega^+ \quad \hat{\theta}_{ij} < T_{ij} \tag{3.8a}$$

$$\forall\, (i,j) \in \omega^- \quad \hat{\theta}_{ij} > t_{ij} \tag{3.8b}$$

The *cost* of a cut ω is

$$Cost(\omega) = \sum_{(i,j)\in\omega^+} c_{ij} - \sum_{(i,j)\in\omega^-} c_{ij}, \tag{3.9}$$

and its *mean-cost* is equal to the cost divided by its cardinality. We call the residual cuts ω_1 and ω_2 to be ***arc-disjoint*** if $\omega_1^+ \cap \omega_2^+ = \emptyset$ and $\omega_1^- \cap \omega_2^- = \emptyset$. Note that arc-disjoint residual cuts might have common arcs such that $(i,j) \in \omega_1^+ \cap \omega_2^-$ or vice versa. In this case, $t_{ij} < \hat{\theta}_{ij} < T_{ij}$ holds for these arcs.

Similar to network flow problems, there are two different alternatives of characterizing optimality conditions for minimum cost tensions. The first one employs combinatorial arguments and cuts whereas the second one uses linear programming duality. Here, we provide both of these optimality conditions without validity proof and refer to the original research papers for details.

Theorem 3.10. *(Hamacher, 1985) A tension $\hat{\theta}$ is optimal if and only if all the residual cuts in G have nonnegative costs.*

Tensions are duals of circulations. Hence, we can also characterize an optimal tension to the minimum cost tension problem using circulations (Pla, 1971).

Theorem 3.11. *A tension $\hat{\theta}$ is a minimum cost tension if and only if there exists a circulation φ on graph G such that*

$$c_{ij} - \varphi_{ij} \geq 0 \qquad \textit{if} \qquad \hat{\theta}_{ij} = t_{ij}, \tag{3.10a}$$

$$c_{ij} - \varphi_{ij} = 0 \qquad \textit{if} \qquad t_{ij} < \hat{\theta}_{ij} < T_{ij}, \tag{3.10b}$$

$$c_{ij} - \varphi_{ij} \leq 0 \qquad \textit{if} \qquad \hat{\theta}_{ij} = T_{ij}. \tag{3.10c}$$

One obvious way of formulating the cost inverse minimum cost tension problem under unit weight ℓ_1-norm is by deriving a linear programming formulation of the inverse problem using the ideas of Ahuja and Orlin (2001) (see Section 1.2) and Theorem 3.11. However, as previously mentioned, our aim is to extend the combinatorial results of network flows (Ahuja and Orlin, 2002) to tensions. Therefore, we will employ mainly Theorem 3.10.

Since the given tension $\hat{\theta}$ is a nonoptimal solution, there exist residual cuts with negative costs in graph G with respect to $\hat{\theta}$. In order to make $\hat{\theta}$ an optimum solution, we have to get rid of these negative cost residual cuts by perturbing the arc costs.

Consider a residual cut ω with a negative cost $Cost(\omega)$. The minimum total perturbation of the costs of the arcs in ω should be $-Cost(\omega)$ so that the cut will have a cost of 0. Using this intuitive idea we can show that the total perturbation, which is needed to make $\hat{\theta}$ an optimum solution, is equal to $-Cost(\Omega^*)$ where $\Omega^* = \{\omega_1^*, \omega_2^*, \ldots, \omega_K^*\}$ denotes a minimum cost collection of arc-disjoint residual cuts in G with respect to $\hat{\theta}$ and

$$Cost(\Omega^*) = \sum_{k=1}^{K} Cost(\omega_k^*).$$

In order to prove this claim, which is restated in Theorem 3.13, we have to first show the following property for a minimum cost collection of arc-disjoint residual cuts in G. This property arises from the duality relationship between tensions and circulations.

Property 3.12. *Let $\Omega^* = \{\omega_1^*, \omega_2^*, \ldots, \omega_K^*\}$ be a minimum cost collection of arc-disjoint residual cuts in G with respect to $\hat{\theta}$. There exists a circulation φ for G such that*

$$\text{if } (i,j) \in \Omega^* \quad : \qquad c_{ij} - \varphi_{ij} \begin{cases} \leq 0 & \text{for } (i,j) \in \Omega^{*+}, \\ \geq 0 & \text{for } (i,j) \in \Omega^{*-}, \end{cases} \tag{3.11a}$$

$$\text{if } (i,j) \notin \Omega^* \quad : \qquad c_{ij} - \varphi_{ij} \begin{cases} \geq 0 & \text{for } \hat{\theta}_{ij} < T_{ij}, \\ \leq 0 & \text{for } \hat{\theta}_{ij} > t_{ij}. \end{cases} \tag{3.11b}$$

Proof: Suppose that there exists a residual cut $\omega_k^* \in \Omega^*$ for which inequalities (3.11a) do not hold i.e., $c_{ij} - \varphi_{ij} \geq 0$ for all $(i,j) \in \omega_k^{*+}$ and $c_{ij} - \varphi_{ij} \leq 0$ for all $(i,j) \in \omega_k^{*-}$. Then,

$$\sum_{(i,j)\in\omega_k^{*+}} (c_{ij} - \varphi_{ij}) - \sum_{(i,j)\in\omega_k^{*-}} (c_{ij} - \varphi_{ij}) \geq 0$$

If we rearrange the inequality as

$$\sum_{(i,j)\in\omega_k^{*+}} c_{ij} - \sum_{(i,j)\in\omega_k^{*-}} c_{ij} - \left(\sum_{(i,j)\in\omega_k^{*+}} \varphi_{ij} - \sum_{(i,j)\in\omega_k^{*-}} \varphi_{ij} \right)$$

and use the fact that the sum of the circulations on cuts is equal to 0, we come up with a contradiction that $\sum_{(i,j)\in\omega_k^{*+}} c_{ij} - \sum_{(i,j)\in\omega_k^{*-}} c_{ij} = Cost(\omega_k^*) \geq 0$. Hence, at least some of the arcs of ω_k^* must satisfy (3.11a).

Now assume that all of the arcs on ω_k^* satisfy (3.11a) except one arc. Without loss of generality we assume that $c_{kl} - \varphi_{kl} > 0$ for $(k,l) \in \omega_k^{*+}$. By definition of the circulation φ, the following inequality holds for ω_k^* :

$$\sum_{(i,j)\in\omega_k^{*+}\setminus\{(k,l)\}} (c_{ij} - \varphi_{ij}) - \sum_{(i,j)\in\omega_k^{*-}} (c_{ij} - \varphi_{ij}) + (c_{kl} - \varphi_{kl}) \leq (c_{kl} - \varphi_{kl}) \tag{3.12}$$

Moreover, the left-hand side of this inequality equals the cost of ω_k^* since the circulations on cuts are 0. Moreover, we know that $\hat{\theta}$ is not a minimum cost tension. Therefore, all the cuts in a minimum cost collection of residual cuts have nonpositive costs, i.e.,

$$Cost(\omega_k^*) \leq 0 < (c_{kl} - \varphi_{kl}),$$

and the inequality in (3.12) is strict.

Now we will construct a new circulation φ^* on G, for which ω_k^* satisfies (3.11a). Consider $\varphi_{kl}^* := c_{kl}$. In order that the circulation φ^* satisfies the zero-balance on cut ω_k^*, we need to set either $\varphi_{ij}^* < \varphi_{ij}$ for an arc $(i,j) \in \omega_k^{*+}$ or assign $\varphi_{ij}^* > \varphi_{ij}$ for an arc $(i,j) \in \omega_k^{*-}$. Since the inequality in (3.12) is strict, there exists either an arc $(i,j) \in \omega_k^{*+}$ such that $c_{ij} - \varphi_{ij} < 0$ or an arc $(i,j) \in \omega_k^{*-}$ such that $c_{ij} - \varphi_{ij} > 0$. Then, we can set $c_{ij} \leq \varphi_{ij}^* < \varphi_{ij}$ for $(i,j) \in \omega_k^{*+}$ or $c_{ij} \geq \varphi_{ij}^* > \varphi_{ij}$ for $(i,j) \in \omega_k^{*-}$. In either case, we can find a circulation φ^* fulfilling the zero-balance on ω_k^* and the conditions (3.11a) for the cut ω_k^*. Furthermore, since the residual cuts in Ω^* are arc-disjoint, we can iteratively apply this modification to construct a circulation satisfying (3.11a-3.11b). Hence, the property holds.

■

Theorem 3.13. *Suppose $\Omega^* = \{\omega_1^*, \omega_2^*, \ldots, \omega_K^*\}$ denotes a minimum cost collection of arc-disjoint residual cuts in G with respect to a given feasible tension $\hat{\theta}$ that is not optimum. Let $Cost(\Omega^*)$ be its cost, which is equal to the total costs of the residual cuts in Ω^*. Then, $-Cost(\Omega^*)$ is the optimal objective function value for the cost inverse minimum cost tension problem under unit weight rectilinear (ℓ_1) norm.*

Proof: Suppose that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$ denotes any collection of arc-disjoint residual cuts with negative costs in G and let c^* denote the optimum cost vector for the inverse problem. First, we show that for the common arcs (i,j) of $\omega_k \in \Omega$, $c_{ij}^* = c_{ij}$ holds.

We know that by Theorem 3.10, the costs of the arcs in $\omega_k \in \Omega$ have to be changed so that $Cost(\omega_k) \geq 0$. Since c^* is the optimum modified cost vector, the following holds (otherwise we could find a cost vector $\bar{c}$ with $\|\bar{c} - c\|_1 \leq \|c^* - c\|_1$).

$$c_{ij}^* \geq c_{ij} \text{ for } (i,j) \in \omega_k^+ \tag{3.13}$$

$$c_{ij}^* \leq c_{ij} \text{ for } (i,j) \in \omega_k^- \tag{3.14}$$

For an arc (i,j) with $(i,j) \in \omega_{k_1}^+, \omega_{k_2}^-$, the inequalities (3.13) and (3.14) must hold with equality. By using this fact, we can show that $-Cost(\Omega)$ is a lower bound on $\|c^* - c\|_1$.

$$\|c^* - c\|_1 = \sum_{(i,j)\in A} |c^*_{ij} - c_{ij}| \geq \sum_{k=1}^{K} \sum_{(i,j)\in\omega_k} |c^*_{ij} - c_{ij}|$$
$$\geq -\sum_{k=1}^{K} Cost(\omega_k) = -Cost(\Omega)$$

Now, we will prove that this lower bound is actually achievable for a minimum cost collection of arc-disjoint residual cuts $\Omega^* = \{\omega^*_1, \omega^*_2, \ldots, \omega^*_K\}$.

Let us choose a circulation φ on G satisfying Property 3.12. Let $c^*_{ij} = \varphi_{ij}$ for $(i,j) \in \Omega^*$ and $c^*_{ij} = c_{ij}$ otherwise. Clearly, this cost vector satisfies the optimality conditions (3.10a - 3.10c), hence it is a feasible solution to the inverse problem. Moreover,

$$\begin{aligned}
\|c^* - c\|_1 &= \sum_{(i,j)\in\Omega^{*-}} (c_{ij} - \varphi_{ij}) - \sum_{(i,j)\in\Omega^{*+}} (c_{ij} - \varphi_{ij}) \\
&= \sum_{k=1}^{K} \sum_{(i,j)\in\omega^{*-}_k} (c_{ij} - \varphi_{ij}) - \sum_{(i,j)\in\omega^{*+}_k} (c_{ij} - \varphi_{ij}) \\
&= -\sum_{k=1}^{K} \left(\sum_{(i,j)\in\omega^{*+}_k} c_{ij} - \sum_{(i,j)\in\omega^{*-}_k} c_{ij} \right) \\
&= -Cost(\Omega^*)
\end{aligned}$$

Thus, the result of the theorem follows. ∎

Notice that in the proof of Theorem 3.13, we use the circulation satisfying Property 3.12 in order to define the new cost vector c^* for which the given tension $\hat{\theta}$ is optimum. We know, by Theorem 3.11, that if a given tension $\hat{\theta}$ is not an optimum solution to the minimum cost tension problem, then there does not exist a feasible circulation on G satisfying the capacity constraints. Since the flow capacities for the circulation problem on G are defined by the costs of the tension problem, the inverse minimum cost tension problem can be indeed interpreted as an inverse feasibility problem, in which the arc capacities are perturbed so that the circulation problem on G is feasible.

We next show that a minimum cost collection of arc-disjoint residual cuts can be found by solving a minimum cost tension problem. Consider the following linear program:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A} c_{ij}(\pi(j)-\pi(i)) && \text{(3.15a)}\\
\text{subject to} \quad & && \\
& -1 \leq \pi(j)-\pi(i) \leq 1 \quad \text{for} \quad (i,j)\in K, && \text{(3.15b)}\\
& 0 \leq \pi(j)-\pi(i) \leq 1 \quad \text{for} \quad (i,j)\in L, && \text{(3.15c)}\\
& -1 \leq \pi(j)-\pi(i) \leq 0 \quad \text{for} \quad (i,j)\in U, && \text{(3.15d)}\\
& \pi \gtrless 0, &&
\end{aligned}
$$

where

$$
\begin{aligned}
K &:= \{(i,j)\in A : t_{ij} < \hat{\theta}_{ij} < T_{ij}\},\\
L &:= \{(i,j)\in A : \hat{\theta}_{ij} = t_{ij}\},\\
U &:= \{(i,j)\in A : \hat{\theta}_{ij} = T_{ij}\}.
\end{aligned}
$$

Obviously, this LP is the formulation of a minimum cost tension problem with lower and upper bounds on the tensions given by the inequalities (3.15b - 3.15d).

Theorem 3.14. *An optimum solution of the LP given by (3.15a - 3.15d) defines a minimum cost collection of arc-disjoint residual cuts on G with respect to $\hat{\theta}$.*

Proof: As already mentioned, the LP (3.15a - 3.15d) is a formulation of a minimum cost tension problem on graph G. Rockafellar (1984) proved that a bounded and feasible tension polyhedra has integral extreme points if all the lower and upper bounds are integer. Besides, every integral tension can be expressed as the difference of some integral node potentials.

The given LP is bounded and has feasible solutions. Moreover, all the bounds are integral, i.e., $t_{ij}, T_{ij} \in \{0, 1, -1\}$ for all $(i,j) \in G$. Therefore, the extreme points of the polyhedra defined by (3.15b - 3.15d) are all integral and for a basic feasible solution $\theta^*_{ij} = \pi(j) - \pi(i)$ of the LP (3.15a - 3.15d) the following holds:

$$
\begin{aligned}
&\theta^*_{ij} \in \{-1, 0\} \quad \text{for } (i,j) \in U,\\
&\theta^*_{ij} \in \{0, 1\} \quad \text{for } (i,j) \in L,\\
&\theta^*_{ij} \in \{-1, 0, 1\} \quad \text{for } (i,j) \in K.
\end{aligned}
$$

By definition of the sets U, L, and K, these basic feasible solutions clearly correspond to the collections of arc-disjoint residual cuts on G with respect to $\hat{\theta}$. Since the objective function minimizes the total cost, an optimum solution of the LP yields a minimum cost collection of arc-disjoint residual cuts.

■

By using Theorems 3.13 and 3.14, we obtain the following conclusion for the cost inverse minimum cost tension problem under unit weight rectilinear norm. Recall that this result can be verified using the LP approach of Ahuja and Orlin (2001) (see Section 1.2).

Corollary 3.15. *The cost inverse minimum cost tension problem under unit weight ℓ_1-norm can be solved by solving a minimum cost tension problem with unit bounds.*

3.2.2 Chebyshev (ℓ_∞) Norm

Ahuja and Orlin (2002) showed that the inverse minimum cost flow problem under unit weight ℓ_∞-norm can be reduced to solving a minimum mean cycle problem in the residual graph. Similarly we will show that the inverse minimum cost tension problem under Chebyshev norm reduces to solving a minimum mean residual cut problem.

In this section, we are given a minimum cost tension problem (MCT) on a directed graph $G = (N, A)$ with upper and lower bound vectors $(T, t) \in (\mathbb{R}^{|A|}, \mathbb{R}^{|A|})$ and a cost vector $c \in \mathbb{R}^{|A|}$. Let $\hat{\theta}$ be a nonoptimal feasible solution to this MCT. The cost inverse minimum cost tension problem under unit weight ℓ_∞-norm is

$$\min \max_{(i,j)\in A} |\hat{c}_{ij} - c_{ij}| \tag{3.16}$$

subject to

$$\begin{aligned} t_{ij} \leq \hat{\theta}_{ij} \leq T_{ij} \quad & \forall\, (i,j) \in A \\ \hat{\theta} \quad & \text{is a minimum cost tension} \\ & \text{with respect to } \hat{c} \end{aligned}$$

As mentioned in Section 3.2.1, a given tension $\hat{\theta}$ is optimal if and only if the graph does not contain any negative cost residual cuts with respect to $\hat{\theta}$. Since in the inverse problem we are given a nonoptimal tension, the graph contains residual cuts with negative costs. Our aim is to modify the cost vector of the arcs from c to $\hat{c}$ such that none of the residual cuts have negative costs and $\max_{(i,j)\in A} |\hat{c}_{ij} - c_{ij}|$ is minimum.

Let ω^* be a minimum mean (cost) residual cut in G with respect to $\hat{\theta}$, i.e., ω^* is a residual cut with $\mu^* = MCost(\omega^*) = Cost(\omega^*)/|\omega^*|$ is minimum among all residual cuts where $|\omega^*|$ denotes the number of arcs in cut ω^*. We adopt an idea of Hadjiat and Maurras (1997) who define *ϵ-optimality* and show that $\epsilon = -\mu^*$ is the smallest positive real number for which $\hat{\theta}$ is ϵ-optimal.

Definition 3.16. For an $\epsilon \geq 0$, a tension $\hat{\theta}$ is ***ϵ-optimal*** if there exists a circulation φ such that

$$\forall\,(i,j) \in A : \begin{cases} (\hat{\theta}_{ij} < T_{ij}) & \Longrightarrow (\varphi_{ij} \leq c_{ij} + \epsilon) \\ (\hat{\theta}_{ij} > t_{ij}) & \Longrightarrow (\varphi_{ij} \geq c_{ij} - \epsilon) \end{cases} \tag{3.17}$$

Theorem 3.17. *(Hadjiat and Maurras, 1997) A tension $\hat{\theta}$ is ϵ-optimal if and only if every cut ω residual with respect to $\hat{\theta}$ satisfies $MCost(\omega) \geq -\epsilon$.*

The definition of ϵ-optimality (3.17) and the previous results imply the following property of the tensions:

Property 3.18. *Let ω^* be a minimum mean residual cut in G with respect to $\hat{\theta}$ and μ^* be the mean cost of it. There exists a circulation φ such that $c_{ij} - \varphi_{ij} = \mu^*$ for the outgoing and $c_{ij} - \varphi_{ij} = -\mu^*$ for the incoming arcs of the cut ω^*. All other arcs satisfy $c_{ij} - \varphi_{ij} \geq \mu^*$ if $\hat{\theta}_{ij} < T_{ij}$ and $c_{ij} - \varphi_{ij} \leq -\mu^*$ if $\hat{\theta}_{ij} > t_{ij}$.*

Theorem 3.19. *Let μ^* denote the mean cost of a minimum mean residual cut in G with respect to a given feasible tension $\hat{\theta}$ that is not optimum. Then, the optimal objective function value for the inverse minimum cost tension problem under unit weight ℓ_∞-norm is $max(0, -\mu^*)$.*

Proof: We choose φ as in Property 3.18. If $\mu^* \geq 0$, then $\hat{\theta}$ is an optimum tension and the theorem is true. Suppose that $\mu^* < 0$ and ω^* is the minimum mean residual cut in G with respect to $\hat{\theta}$. Let z^* be the optimum solution to the inverse minimum tension problem under Chebyshev norm. We first claim that $z^* \geq -\mu^*$. Recall

$$Cost(\omega^*) = \sum_{(i,j)\in\omega^{*+}} c_{ij} - \sum_{(i,j)\in\omega^{*-}} c_{ij} = |\omega^*|\mu^*$$

If $z^* < -\mu^*$, then, in order to make $\hat{\theta}$ the optimal solution, it would be sufficient to increase the costs of $(i,j) \in \omega^{*+}$ by an amount z^* and decrease the costs of $(i,j) \in \omega^{*-}$ by z^*. The resulting cost of the cut ω^* is $|\omega^*|\mu^* + |\omega^*|z^* < 0$, which is a contradiction to the optimality of $\hat{\theta}$. Hence, $z^* \geq -\mu^*$.

Now we prove that there exists a vector c^* with $\|c^* - c\| = -\mu^*$ such that $\hat{\theta}$ is optimal with respect to c^*. Define c^* as follows:

$$c^*_{ij} = \begin{cases} c_{ij} - \mu^* & \text{if } \hat{\theta}_{ij} < T_{ij} \text{ and } c_{ij} - \varphi_{ij} < 0 \\ c_{ij} + \mu^* & \text{if } \hat{\theta}_{ij} > t_{ij} \text{ and } c_{ij} - \varphi_{ij} > 0 \\ c_{ij} & \text{otherwise} \end{cases} \tag{3.18}$$

It is obvious that $\|c^* - c\|_\infty \leq -\mu^*$. Moreover, by Property 3.18

$$\begin{aligned} c^*_{ij} - \varphi_{ij} = c_{ij} - \mu^* - \varphi_{ij} &\geq \mu^* - \mu^* = 0 \qquad \text{for } \hat{\theta}_{ij} < T_{ij} \\ c^*_{ij} - \varphi_{ij} = c_{ij} + \mu^* - \varphi_{ij} &\leq \mu^* - \mu^* = 0 \qquad \text{for } \hat{\theta}_{ij} > t_{ij} \end{aligned}$$

Hence, $\hat{\theta}$ satisfies the optimality conditions and c^* is an optimal solution of the inverse minimum cost tension problem under Chebyshev norm.

■

Hadjiat and Maurras (1997) provide a Newton type algorithm to solve the minimum mean residual cut problem. Using their algorithm we can find an optimum solution for the inverse problem in strongly polynomial time. McCormick and Ervolina (1994) study max mean weight cuts and mention that a direct method of calculating max mean weight cuts as Karp (1972) does for minimum mean cycles has not yet been found. Radzik (1993) improves the best known running time bound of Newton's method for maximum mean weight cut problem and proves that Newton's method runs in a strongly polynomial number of iterations for all linear fractional optimization problems. He also shows that the maximum mean weight cut problem, *parametric flow problem* and *minimum maximum arc cost flow problem* are closely related to each other. Here, we revise Radzik's result (Radzik, 1993) to include the inverse minimum cost tension problem under Chebyshev distance.

An instance of the parametric flow problem (PF) consists of a network G with arc capacities u and supplies/demands on nodes, and a weight function $w : A \to \mathbb{R}^{|A|}$. The goal is to find minimum nonnegative δ such that $G_{u+w\delta}$, network G with capacity function $u + w\delta$, is feasible. Minimum maximum arc cost flow problem (MMAC) is defined on a network G with a nonnegative cost function $c : A \to \mathbb{R}^{|A|}$. The goal is to find a flow f satisfying the demands on nodes while minimizing the maximum arc cost i.e., minimizing $\max_{(i,j)\in A} c_{ij} f_{ij}$. In the uniform versions of the problems all weights and costs are equal to 1, respectively.

The relationship between IMCT$_c$ under ℓ_∞-norm and PF is more straightforward to justify. In IMCT$_c$, we are given a tension $\hat{\theta}$, which is feasible to MCT with cost vector c but not optimal. Hence, the dual circulation problem of the given MCT problem is infeasible, i.e., there does not exist a circulation φ to satisfy Theorem 3.11. Our aim is to find the minimum $\mu \geq 0$ such that the circulation problem on G with arc capacities $c + w\mu$ is feasible. In this case $w_{ij} = 1$ if $\hat{\theta}_{ij} < T_{ij}$ and $w_{ij} = -1$ if $\hat{\theta}_{ij} > t_{ij}$.

In order to show the relationship between IMCT$_c$ under ℓ_∞-norm and MMAC problem we exploit LP duality. We apply the linear programming method of Ahuja and Orlin (2001) to obtain the following LP formulation for IMCT$_c$ under unit weight ℓ_∞-norm.

$$\text{Minimize} \quad \sum_{(i,j)\in A} c_{ij}(\pi(j) - \pi(i)) \tag{3.19}$$

subject to

$$\begin{aligned}
\sum_{(i,j)\in A} \eta_{ij} &= 1 \\
-\eta_{ij} \leq \pi(j) - \pi(i) &\leq \eta_{ij} \quad \text{for } (i,j) \in K \\
0 \leq \pi(j) - \pi(i) &\leq \eta_{ij} \quad \text{for } (i,j) \in L \\
-\eta_{ij} \leq \pi(j) - \pi(i) &\leq 0 \quad \text{for } (i,j) \in U \\
\eta \geq 0 \quad \pi &\gtreqless 0
\end{aligned}$$

By Theorem 3.19 we know that (3.19) is the LP formulation for finding a minimum mean cost residual cut in G with respect to $\hat{\theta}$. Let us consider its dual.

$$\begin{aligned} &\text{Maximize} \quad \lambda && (3.20)\\ &\text{subject to} \\ \sum_{j\in N^+(i)} (\varphi^1_{ij} - \varphi^2_{ij}) - \sum_{j\in N^-(i)} (\varphi^1_{ji} - \varphi^2_{ji}) &= \sum_{j\in N^-(i)} c_{ji} - \sum_{j\in N^+(i)} c_{ij} && \forall\, i \in N \\ \lambda &\leq -(\varphi^1_{ij} + \varphi^2_{ij}) && \forall\, (i,j) \in K \\ \lambda &\leq -(\varphi^1_{ij}) && \forall\, (i,j) \in L \\ \lambda &\leq -(\varphi^2_{ij}) && \forall\, (i,j) \in U \\ \varphi^1_{ij}, \varphi^2_{ij} &\geq 0 \end{aligned}$$

Obviously, (3.20) is an instance of the MMAC problem on a graph $G' = (N, A')$ with

$$A' := \{(i,j) : (i,j) \in A\} \cup \{(j,i) : (i,j) \in A\}.$$

The demands/supplies on the nodes are

$$\sum_{j\in N^-(i)} c_{ji} - \sum_{j\in N^+(i)} c_{ij} = -Cost(\omega(i)) \quad \forall\, i \in N$$

and the flow capacities of the arcs are $[0, \infty)$. The costs of the arcs are

$$c_{ij} = \begin{cases} 1 & \text{for } (i,j) \in K \cup L, \\ 1 & \text{for } (j,i) \in K \cup U, \\ 0 & \text{for } (j,i) \in L, \\ 0 & \text{for } (i,j) \in U. \end{cases}$$

This result establishes the fact that IMCT$_c$ under ℓ_∞-norm and MMAC problems are equivalent to each other.

If we are given positive weights $w_{ij} > 0 \quad \forall\, (i,j) \in A$, the objective function of IMCT$_c$ under Chebyshev distance would be

$$\text{Minimize} \quad \max_{(i,j)\in A} w_{ij}(|c_{ij} - \hat{c}_{ij}|) \tag{3.21}$$

In this case, the inverse problem reduces to finding a minimum mean-weight residual cut on graph G.

Ahuja and Orlin (2002) showed, using combinatorial arguments, that the cost inverse minimum cost flow problem under weighted ℓ_∞-norm reduces to solving a minimum cost-to-weight ratio cycle problem on the residual graph. Here, we briefly repeat their combinatorial arguments for tensions.

Consider a negative cost residual cut ω in G with respect to $\hat{\theta}$. In the optimal solution of the inverse problem the total perturbation of the arc costs on ω should be at least $-Cost(\omega)$ in order to eliminate this negative cost residual cut. Suppose that we increase, respectively decrease, the cost of an arc $(i,j) \in \omega^+$, respectively $(i,j) \in \omega^-$, by α_{ij}. Then, $\sum_{(i,j)\in\omega} \alpha_{ij} \geq -Cost(\omega)$ and the impact of this change on the objective function is $\max\{w_{ij}\alpha_{ij} : (i,j) \in \omega\}$. This impact is minimum when $w_{ij}\alpha_{ij} = M$ for all $(i,j) \in \omega$ and for some $M \in \mathbb{R}_+$. Since the total change must be at least $-Cost(\omega)$, we can compute that

$$M \geq \frac{-Cost(\omega)}{\sum_{(i,j)\in\omega} \frac{1}{w_{ij}}}.$$

Consequently, the cost inverse minimum cost tension problem under weighted ℓ_∞-norm can be solved by finding a minimum mean-weight residual cut with the arc weights $\tau_{ij} := \frac{1}{w_{ij}}$ for all $(i,j) \in A$.

3.3 Capacity Inverse Minimum Cost Tension Problem

In Section 2.2 we analyzed the capacity inverse minimum cost flow problem and showed that the problem is $\mathcal{NP}$-complete under rectilinear norm whereas a greedy algorithm provides in polynomial time an optimal solution for the Chebyshev norm. In this section we will analyze an analogous problem, namely the *capacity inverse minimum cost tension problem* under Chebyshev distance and show that a modified version of the greedy algorithm of Section 2.2.3 solves this problem, as well. Obviously, it is an important research study to analyze the capacity inverse minimum cost tension problem also under rectilinear norm. However, since the same problem is $\mathcal{NP}$-hard for the flow case, we did not attempt, in this thesis, to carry this analysis over to tensions. We believe that it is first necessary to make a thorough analysis of the polyhedral description of the flow case and develop efficient algorithms to solve it. Such a detailed analysis of this problem is beyond the scope of this thesis and is, therefore, left for future research.

In Section 3.3.1 we define the capacity inverse minimum cost tension problem and investigate its feasibility. In Section 3.3.2 we prove that the greedy algorithm of Section 2.2.3 solves the problem under Chebyshev norm.

3.3.1 Problem Definition

Similar to the previous sections, we are given a connected digraph $G = (N, A)$ and a minimum cost tension problem defined on G with cost vector c and bound vectors (t, T). Moreover, we have a feasible tension $\hat{\theta}$, which is not an optimum solution for the given minimum cost tension problem. The aim of the capacity inverse minimum cost tension problem (IMCT$_u$) is to perturb the lower and upper bound vectors from (t, T) to $(\hat{t}, \hat{T})$ such that $\hat{\theta}$ is a minimum cost tension with respect to the new bounds

$(\hat{t}, \hat{T})$ and the perturbation $\|(t,T)-(\hat{t},\hat{T})\|$ is minimum according to some norm. Consequently, the capacity inverse minimum cost tension problem can be formulated as

$$\begin{aligned} \min \quad & \|(t,T)-(\hat{t},\hat{T})\| && (3.22a) \\ \text{subject to} \quad & && \\ & \hat{t}_{ij} \leq \hat{\theta}_{ij} \leq \hat{T}_{ij} \quad \forall (i,j) \in A, && (3.22b) \\ & \hat{\theta} \text{ optimal min cost tension} && (3.22c) \\ & \text{with respect to bounds } (\hat{t},\hat{T}). && \end{aligned}$$

Recall that $\hat{\theta}$ and (t,T) are part of the data while $(\hat{t},\hat{T})$ is the vector of variables to be determined.

By Theorem 3.10 it is known that the graph G contains negative cost residual cuts with respect to $\hat{\theta}$ since it is not an optimal tension. Therefore, we can formulate the following lemma:

Lemma 3.20. *The feasible solutions $(\hat{t},\hat{T})$ of the capacity inverse minimum cost tension problem satisfy the following:*

- $t_{ij} \leq \hat{t}_{ij} \leq \hat{T}_{ij} \leq T_{ij}$,
- *if* $t_{ij} < \hat{t}_{ij}$, *then* $\hat{t}_{ij} = \hat{\theta}_{ij}$,
- *if* $\hat{T}_{ij} < T_{ij}$, *then* $\hat{T}_{ij} = \hat{\theta}_{ij}$.

Proof: Since $\hat{\theta}$ is not an optimal tension, there exist negative cost residual cuts with respect to $\hat{\theta}$. Hence, we can interpret the capacity inverse minimum cost tension problem as destroying the negative cost residual cuts by changing the arc bounds but respecting the feasibiliy of the given tension $\hat{\theta}$.

By definition of residual cuts, the arcs in ω^+ have $\hat{\theta}_{ij} < T_{ij}$ and the arcs in ω^- have $\hat{\theta}_{ij} > t_{ij}$, where ω is a residual cut with respect to $\hat{\theta}$. Thus, it is enough to either reduce an upper bound from T_{ij} to $\hat{\theta}_{ij}$ for an arc in ω^+ or increase a lower bound from t_{ij} to $\hat{\theta}_{ij}$ in order to eliminate a negative cost residual cut. Consequently, the claim of the lemma is true.

■

Lemma 3.21. *There always exists a feasible solution for the capacity inverse minimum cost tension problem.*

Proof: Consider the lower and upper bounds $(\hat{t},\hat{T})$ with $\hat{t}_{ij} = \hat{T}_{ij} = \hat{\theta}_{ij}$ for all $(i,j) \in A$. Obviously, bounds $(\hat{t},\hat{T})$ fulfill the conditions of Lemma 3.20. Moreover, the given tension $\hat{\theta}$ is the only feasible solution to the minimum cost tension problem on G with respect to the arcs bounds $(\hat{t},\hat{T})$, and hence, it is the only optimum solution to the minimum cost tension problem. As a result, the vector of the lower and upper bounds $(\hat{t},\hat{T})$ is a feasible solution to the capacity inverse minimum cost tension problem.

3.3.2 Chebyshev (ℓ_∞) Norm

In this section, we show that the capacity inverse minimum cost tension problem under ℓ_∞-norm (abbreviated subsequently as C-IMCT$_u$) is polynomially solvable using a modified version of the greedy algorithm of the flow case in Section 2.2.3. The objective function of C-IMCT$_u$ is

$$\min \max_{(i,j)\in A} \{\max\{|\hat{T}_{ij} - T_{ij}|, |\hat{t}_{ij} - t_{ij}|\}\}. \tag{3.23}$$

At each iteration of the greedy algorithm we select a negative cost residual cut by using the algorithm of Hadjiat and Maurras (1997) and modify the upper or lower bound of an arc which is in the cut. For the bound modification we choose an arc $(i^*, j^*) \in \omega$ with

$$(i^*, j^*) = \arg\min\{\min_{(i,j)\in\omega^+} (T_{ij} - \hat{\theta}_{ij}), \min_{(i,j)\in\omega^-} (\hat{\theta}_{ij} - t_{ij})\}. \tag{3.24}$$

Algorithm 10. (Greedy Algorithm for Tensions)

1. Initialize $(\hat{t}, \hat{T}) = (t, T)$.

2. Find a minimum cost residual cut ω using the algorithm described by Hadjiat and Maurras (1997).
   IF the cost of the cut is nonnegative, STOP and Output: $(\hat{t}, \hat{T})$.
   ELSE GO TO Step-3.

3. Find an arc $(i^*, j^*) \in \omega$ satisfying the equation (3.24).
   Then set $\hat{T}_{i^*j^*} = \hat{\theta}_{i^*j^*}$ if $(i^*, j^*) \in \omega^+$ or set $\hat{t}_{i^*j^*} = \hat{\theta}_{i^*j^*}$ if $(i^*, j^*) \in \omega^-$.
   GO TO Step-2.

Theorem 3.22. *The greedy algorithm solves the capacity inverse minimum cost tension problem under unit weight ℓ_∞-norm optimally.*

Proof: To prove the correctness of the algorithm, we assume that (t^*, T^*) is an optimal solution of the capacity inverse minimum cost tension problem under unit weight ℓ_∞-norm and $(t^*, T^*) \neq (\hat{t}, \hat{T})$ where $(\hat{t}, \hat{T})$ is the solution delivered by the greedy algorithm. Hence,

$$\max_{(i,j)\in A} \{\max\{|T^*_{ij} - T_{ij}|, |t^*_{ij} - t_{ij}|\}\} \leq \max_{(i,j)\in A} \{\max\{|\hat{T}_{ij} - T_{ij}|, |\hat{t}_{ij} - t_{ij}|\}\}.$$

Moreover, by construction of the greedy algorithm there exists a negative cost residual cut ω for which

$$\arg \max_{(i,j)\in A} \{\max\{|\hat{T}_{ij} - T_{ij}|, |\hat{t}_{ij} - t_{ij}|\}\} =: (i^*, j^*) \in \omega$$

and

$$(i^*, j^*) = \arg\min\{\min_{(i,j)\in\omega^+}(T_{ij} - \hat{\theta}_{ij}), \min_{(i,j)\in\omega^-}(\hat{\theta}_{ij} - t_{ij})\}.$$

Then, $\forall\ (i,j) \in \omega$,

$$\begin{aligned}\max_{(i,j)\in A}\{\max\{|T^*_{ij} - T_{ij}|, |t^*_{ij} - t_{ij}|\}\} &\leq \max_{(i,j)\in A}\{\max\{|\hat{T}_{ij} - T_{ij}|, |\hat{t}_{ij} - t_{ij}|\}\} \\ &\leq \min\{\min_{(i,j)\in\omega^+}(T_{ij} - \hat{\theta}_{ij}), \min_{(i,j)\in\omega^-}(\hat{\theta}_{ij} - t_{ij})\}\end{aligned}$$

holds. We also conclude from Lemma 3.20 that at least one arc from each negative cost residual cut has a modified bound. So, there exists an arc $(k,l) \in \omega$ with $T^*_{kl} \neq T_{kl}$ if $(k,l) \in \omega^+$ or $t^*_{kl} \neq t_{kl}$ if $(k,l) \in \omega^-$. Without loss of generality assume that $(k,l) \in \omega^+$. Then,

$$\begin{aligned}T_{kl} - T^*_{kl} \leq \max\{|T^*_{kl} - T_{kl}|, |t^*_{kl} - t_{kl}|\} &\leq \max_{(i,j)\in A}\{\max\{|T^*_{ij} - T_{ij}|, |t^*_{ij} - t_{ij}|\}\} \\ &\leq \max_{(i,j)\in A}\{\max\{|\hat{T}_{ij} - T_{ij}|, |\hat{t}_{ij} - t_{ij}|\}\} \\ &\leq \min\{\min_{(i,j)\in\omega^+}(T_{ij} - \hat{\theta}_{ij}), \min_{(i,j)\in\omega^-}(\hat{\theta}_{ij} - t_{ij})\} \\ &\leq T_{kl} - \hat{\theta}_{kl}\end{aligned}$$

Since (t^*, T^*) has to satisfy Lemma 3.20, all the inequalities hold with equality and the solution of the greedy algorithm is optimum.

■

Before we proceed to compute the running time of the greedy algorithm for the tension case, we briefly revise the minimum cost residual cut algorithm of Hadjiat and Maurras (1997). The authors first associate two different costs (c^+_{ij} and c^-_{ij}) to each arc of the graph in such a way that the costs of all the residual cuts remain the same whereas the cost of a nonresidual cut is greater than any residual one. These new costs are defined as follows:

- if $t_{ij} < \hat{\theta}_{ij} < T_{ij}$, then $c^+_{ij} = c^-_{ij} = c_{ij}$;
- if $t_{ij} = \hat{\theta}_{ij} < T_{ij}$, then $c^+_{ij} = c_{ij}$ and $c^-_{ij} = -M$;
- if $t_{ij} < \hat{\theta}_{ij} = T_{ij}$, then $c^+_{ij} = M$ and $c^-_{ij} = c_{ij}$;

where M is a sufficiently large number.

Moreover, the authors redefine the cost of a cut ω as

$$Cost(\omega) = \sum_{(i,j)\in\omega^+} c^+_{ij} - \sum_{(i,j)\in\omega^-} c^-_{ij}.$$

Then, they prove that the redefined cost function $Cost(\omega(V))$ of a cut induced by a subset of vertices $V \subseteq N$, which is defined with respect to the new cost vectors, is submodular, i.e., $\forall\, V_1, V_2 \subseteq N$,

$$Cost(\omega(V_1)) + Cost(\omega(V_2)) \geq Cost(\omega(V_1 \cup V_2)) + Cost(\omega(V_1 \cap V_2)).$$

Moreover, for disjoint subsets V_1, V_2, V_3 of N, the authors show that this cost function has the following property:

$$\begin{aligned} Cost(\omega(V_1 \cup V_2 \cup V_3)) = & Cost(\omega(V_1 \cup V_2)) + Cost(\omega(V_1 \cup V_3)) + Cost(\omega(V_2 \cup V_3)) \\ & - Cost(\omega(V_1)) - Cost(\omega(V_2)) - Cost(\omega(V_3)). \end{aligned}$$

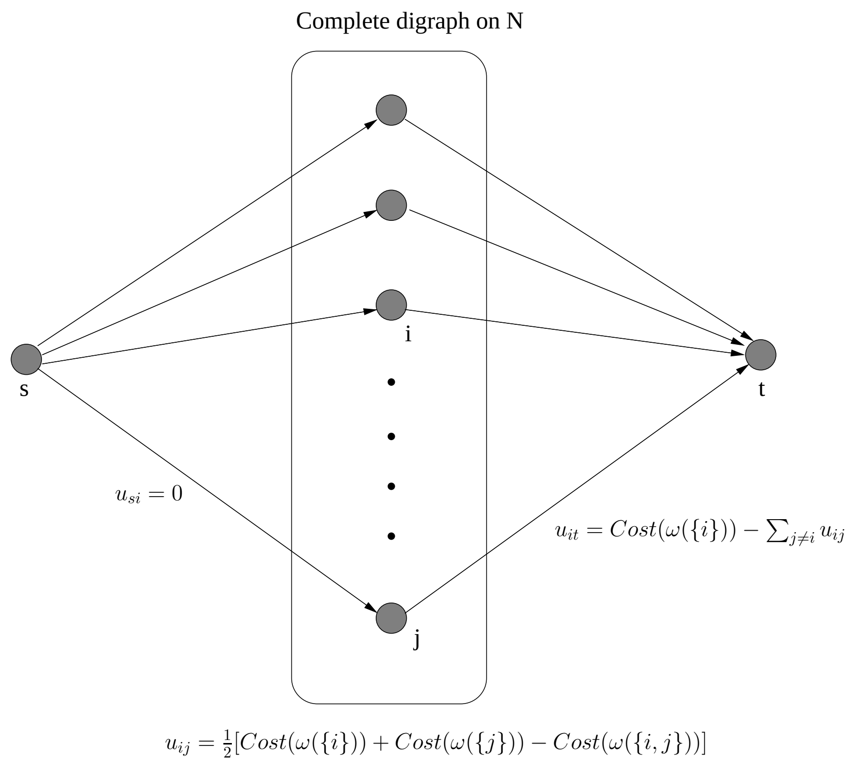

Figure 3.1: Graph $G' = (N', A')$ from the algorithm of Hadjiat and Maurras (1997)

Consequently, they can apply the results of Cunningham (1985) and show that the minimum cost residual cut problem can be solved as a minimum $s-t$ cut problem on a graph $G' = (N', A')$ where $N' = N \cup \{s, t\}$ and

$$A' = \{(s,i) : i \in N\} \cup \{(i,t) : i \in N\} \cup \{(i,j) \in N \times N : i \neq j\}.$$

The capacities u_{ij} of the arcs are defined as

$$u_{ij} = \begin{cases} 0 & \text{if } i = s, \\ \frac{1}{2}[Cost(\omega(\{i\})) + Cost(\omega(\{j\})) - Cost(\omega(\{i,j\}))] & \text{if } (i,j) \in N \times N : i \neq j, \\ Cost(\omega(\{i\})) - \sum_{k \in N \setminus \{i\}} u_{ik} & \text{if } j = t. \end{cases}$$

Since a minimum $s-t$ cut on graph can be computed by a maximum flow calculation, the running time of the minimum cost residual cut algorithm of Hadjiat and Maurras (1997) is $O(n^3)$.

Theorem 3.23. *The worst case running time of the greedy algorithm for the capacity inverse minimum cost tension problem under unit weight Chebyshev norm is* $O(2mn^3)$.

If only gravity were working, the path would be symmetrical, it is the wind resistance that produces the tragic curve.

Norman Mailer (1923 - 2007)

4 Inverse Matroid Flows On Regular Matroids

This chapter analyzes the generalization of inverse network flow and tension problems to matroid flows in regular matroids. As it was already mentioned in Introduction and Outline, we concentrate, in this thesis, on the generalization of some theoretical results of inverse network flows to matroid flows, but do not provide any general solution algorithms or complexity analysis.

In Section 4.3 we investigate inverse maximal M-flow problem under rectilinear and Chebyshev norms. Section 4.4 is dedicated to the analysis of cost inverse minimum cost M-flow problem. In Section 4.5 we present a generalization for capacity inverse problems under Chebyshev norm and show that the greedy algorithm also provides an optimum solution to the capacity inverse minimum cost M-flow problem under Chebyshev norm. The analysis of the capacity inverse problem under rectilinear norm is left uncovered for the same reasons as in the case of tensions.

First, we present a short introduction to Matroid Theory and provide important definitions and results of this field that are required for the study of inverse matroid flows. A more detailed introduction was provided by Bunke (2006). For a more extensive investigation of the subject, we refer to the books of Welsh (1976) and Oxley (1992) as well as the original work of Whitney (1935).

4.1 Introduction to Matroids

A *matroid* is a structure that captures the essence of the notion of independence together with some additional properties. Consider the following basic definition:

Definition 4.1. A *matroid* is a pair $M = (E, \mathfrak{I})$ consisting of a finite *ground set* E with $|E| = m$ and a collection of *independent sets* $\mathfrak{I} \subseteq \mathscr{P}(E)$ satisfying

(I1) $\emptyset \in \mathfrak{I}$,

(I2) if $I \in \mathfrak{I}$ and $I' \subseteq I$ then $I' \in \mathfrak{I}$, and

(I3) if $I_1, I_2 \in \mathfrak{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup \{e\} \in \mathfrak{I}$.

Here, $\mathscr{P}(E)$ is the powerset of the set E.

If only (I1) and (I2) are fulfilled then $(E, \mathfrak{I})$ is called an *independence system*. A subset $X \subseteq E$ with $X \notin \mathfrak{I}$ is called a *dependent set*. A set $C \notin \mathfrak{I}$ with $F \in \mathfrak{I}$ for all $F \subsetneq C$, i.e. a minimal dependent subset of E, is called a *circuit* of M. The collection of circuits of M is denoted by $\mathscr{C}$. For every set $F \subseteq E$, a set $B \subseteq F$ with $B \in \mathfrak{I}$ and $X \notin \mathfrak{I}$ for all X with $B \subsetneq X \subseteq F$ is called a *base* of F, i.e. B is a maximal independent subset of F. It can be shown that all the bases of a set F have the same number of elements. A base of the matroid M is a base of E and the collection of the bases of M is denoted by $\mathscr{B}$.

In Definition 4.1 a matroid on ground set E is specified by its independent sets. However, there exist many different equivalent ways to characterize the matroids. In this chapter we make use of a characterization by the circuits. This approach seems to be the most natural when coming from Graph Theory.

Theorem 4.2. *A set of subsets of a ground set E, $\mathscr{C} \subseteq \mathscr{P}(E)$, is the collection of circuits of a matroid M on E if and only if*

(C1) $\emptyset \notin \mathscr{C}$,

(C2) *if $C_1 \neq C_2$ are distinct circuits, then $C_1 \nsubseteq C_2$, and*

(C3) *if $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) \backslash \{e\}$.*

Then, the collection of independent sets of M is given by $\mathfrak{I} = \{I \subseteq E : C \nsubseteq I, \ \forall C \in \mathscr{C}\}$.

The concept of duality is of fundamental importance in Matroid Theory. It extends the concept of orthogonality in vector spaces as well as the notion of planar duality in plane graphs. The theory of matroid duality was introduced by Whitney (1935) who proved the following basic result.

Theorem 4.3. *Let $M = (E, \mathfrak{I})$ be a matroid and let $\mathscr{B}$ be the set of bases of M. Then, $\mathscr{B}^* = \{E \backslash B : B \in \mathscr{B}\}$ is the set of bases of a matroid M_d on E.*

Definition 4.4. The matroid M_d of Theorem 4.3 is called the *dual matroid* of M. Circuit, base, etc. of the dual matroid M_d are often referred to as *cocircuit*, *cobase*, etc. of M.

From the fact that the bases of M_d are the complements of the bases of M it follows that $(M_d)_d = M$ and every matroid has a dual (Whitney, 1935).

Lemma 4.5. *A subset $D \subseteq E$ is a cocircuit of matroid M if and only if $D \cap B \neq \emptyset$ for every base B of M and D is minimal with this property.*

We denote the set of all cocircuits of a matroid M as $\mathscr{D}$.

As mentioned previously, this chapter is dedicated to an important class of matroids, namely *regular matroids*.

Definition 4.6. If it is possible to partition each $C \in \mathscr{C}$ and $D \in \mathscr{D}$ into $C = C^+ \cup C^-$ and $D = D^+ \cup D^-$ such that

$$|C^+ \cap D^+| + |C^- \cap D^-| = |C^+ \cap D^-| + |C^- \cap D^+| \tag{4.1}$$

holds for each pair of circuits and cocircuits, then M is called a *regular matroid*.

Tutte (1958) showed that a matroid is regular if and only if it can be represented by the columns of a totally unimodular matrix. Recall that a matrix is called *totally unimodular* if its every square submatrix has a determinant of $\{\pm 1, 0\}$. Simple examples for regular matroids are *graphic* and *cographic* matroids.

Theorem 4.7. *Let E be the set of edges of graph G and let $\mathscr{C}$ be the set of edge sets of elementary cycles of G. Then $(E, \mathscr{C})$ is a matroid.*

The matroid derived from graph G as above is called the *cycle matroid* of G and denoted by $M(G)$. Two matroids $M_1 = (E_1, \mathfrak{I}_1)$ and $M_2 = (E_2, \mathfrak{I}_2)$ are called *isomorphic* to each other if there exists a one-to-one correspondence between the elements of E_1 and E_2 that preserves independence.

Definition 4.8. A matroid is called *graphic* if it is isomorphic to the cycle matroid of a graph.

Definition 4.9. The dual of the cycle matroid of a graph G is called the *cocycle matroid* of G and denoted by $M_d(G)$.

Definition 4.10. A matroid is called *cographic* if it is isomorphic to the cocycle matroid of a graph.

Throughout this chapter we use, following Burkard and Hamacher (1981), the notation stated below:

$$\begin{aligned} \mathscr{C}_e &:= \{C \in \mathscr{C} : e \in C\}, & \mathscr{C}_e{}^+ &:= \{C \in \mathscr{C} : e \in C^+\}, \\ \mathscr{C}_e{}^- &:= \{C \in \mathscr{C} : e \in C^-\}, & \mathscr{C}^+ &:= \{C \in \mathscr{C} : C = C^+\}. \end{aligned}$$

$\mathscr{D}_e$, $\mathscr{D}_e{}^+$, $\mathscr{D}_e{}^-$ are defined analogously.

4.2 Flows on Regular Matroids

In this section we review the basic definitions and concepts of matroid flows. Our aim is to underline the main assumptions and results that will be useful in the following sections. For a more detailed analysis of this topic and its generalization to algebraic matroid flows we refer to Hamacher (1980), Burkard and Hamacher (1981), Hamacher (1982a) and Hamacher (1982b).

Matroid flows and circulations which are introduced in this section are functions defined on the ground set E and the circuit set $\mathscr{C}$ of a regular matroid M, respectively.

Definition 4.11. The function $f : E \to \mathbb{R}^{|E|}$ is called a *matroid flow* (or *M-flow*) if it satisfies the following *cocircuit property*

$$f(D^+) := \sum_{e \in D^+} f(e) = \sum_{e \in D^-} f(e) =: f(D^-) \quad \forall\, D \in \mathscr{D}. \tag{4.2}$$

Hamacher (1982a) studied the circuit decomposition of the M-flows and defined a *matroid circulation* as follows:

Definition 4.12. A *matroid circulation* (M-circulation) is any function $g : \mathscr{C} \to \mathbb{R}^{|\mathscr{C}|}$ which is defined on the circuit set $\mathscr{C}$ of M.

Each M-circulation g induces a function $f_g : E \to \mathbb{R}^{|E|}$ defined by

$$f_g(e) = \sum_{C \in \mathscr{C}_e{}^+} g(C) - \sum_{C \in \mathscr{C}_e{}^-} g(C). \tag{4.3}$$

Theorem 4.13. ***(Equivalence Theorem)*** *f is an M-flow on a regular matroid M if and only if there exists an M-circulation g such that $f \equiv f_g$.*

Suppose $\tilde{e} \in E$ is a distinguished element of the ground set and we are given two capacity functions $k : \tilde{E} \to \mathbb{R}^{|\tilde{E}|}$ and $r : \tilde{E} \to \mathbb{R}^{|\tilde{E}|}$ with $\tilde{E} = E \backslash \{\tilde{e}\}$.

Definition 4.14. A function $f : E \to \mathbb{R}^{|E|}$ is called an *admissible M-Flow* if it fulfills the cocircuit property (4.2) and the following *capacity property*

$$r(e) \leq f(e) \leq k(e) \quad \forall\, e \in \tilde{E}. \tag{4.4}$$

Maximal M-flow problem is determining an admissible M-flow f of maximum value $f(\tilde{e})$, a so-called *maximal M-flow*. Similar to the max-flow min-cut and max-tension min-path theorems (Theorems 1.4 and 3.1) one can show a relationship between a maximal M-flow and a minimum capacity cocircuit of the matroid M.

Theorem 4.15. ***(Max M-flow Min Cocircuit Theorem)*** *An admissible M-flow f is a maximal M-flow if and only if there exists a cocircuit $D \in \mathscr{D}_{\tilde{e}}{}^-$ with*

$$f(\tilde{e}) = \sum_{e \in D^+} k(e) - \sum_{e \in D^- \backslash \{\tilde{e}\}} r(e) = k(D^+) - r(D^- \backslash \{\tilde{e}\}). \tag{4.5}$$

A cocircuit satisfying (4.5) is called an *f-saturated cocircuit*. We name the value of $k(D^+) - r(D^- \backslash \{\tilde{e}\})$ the *capacity* of a cocircuit and denote it with $K(D)$. A circuit

$C \in \mathscr{C}_{\tilde{e}}{}^{+}$ is called *f-augmenting*, if $f(e) > r(e)$ for all $e \in C^-$ and $f(e) < k(e)$ for all $e \in C^+ \backslash \{\tilde{e}\}$.

Let $a : E \to \mathbb{R}^{|E|}$ be a *cost function* with $a(\tilde{e}) = 0$. The cost of an M-flow f is defined by

$$A(f) := \sum_{e \in E} f(e)a(e). \tag{4.6}$$

Minimum cost M-flow problem is finding a maximal M-flow f which minimizes $A(f)$. Burkard and Hamacher (1981) use the definition of a *negative circuit* with respect to f to prove the optimality conditions for the minimum cost M-flow problem.

Definition 4.16. A circuit $C \in \mathscr{C}$ with $f(e) > r(e)$ for all $e \in C^-$ and $f(e) < k(e)$ for all $e \in C^+$ and $\tilde{e} \notin C$ is called a *negative circuit* (with respect to f) if

$$a(C) := a(C^+) - a(C^-) < 0,$$

where $a(C^+) = \sum_{e \in C^+} a(e)$ and $a(C^-)$ is defined analogously.

For the sake of simplicity, we will call a circuit with $f(e) > r(e)$ for all $e \in C^-$ and $f(e) < k(e)$ for all $e \in C^+$ and $\tilde{e} \notin C$ a *residual circuit* and denote the set of all such circuits with $\mathscr{C}_R$. Hence, the negative circuits are the residual circuits with negative costs.

Theorem 4.17. ***(Negative Circuit Theorem)*** *Let f be a maximal M-flow on the regular matroid $M = (E, \mathscr{C})$. f is a minimum cost M-flow if and only if there exists no negative circuit with respect to f.*

Burkard and Hamacher (1981) presented two algorithms to solve the minimum cost M-flow problem. The first algorithm, which is called *negative circuit algorithm*, eliminates the negative circuits at each iteration. The *shortest circuit algorithm*, on the other hand, moves from one *extreme M-flow* to some other with larger flow value $f(\tilde{e})$ until an optimum solution is found. An *extreme M-flow* f is an M-flow with

$$A(f) = \min\{A(\tilde{f}) : \tilde{f}(\tilde{e}) = f(\tilde{e})\}.$$

4.3 Inverse Maximal M-Flow Problem

In this section we consider the inverse problem of maximal M-flows. We are given an instance of a maximal M-flow problem on a regular matroid M with capacity functions $k(e) \in \mathbb{R}_+$ and $r(e) = 0 \ \ \forall\, e \in \tilde{E}$. Let $\tilde{f}$ be an admissible M-flow which is not a maximal M-flow. The *inverse maximal M-flow problem* is perturbing the capacities from $k(e)$ to $\tilde{k}(e)$ such that

$$\|w(\tilde{k} - k)\| \tag{4.7}$$

is minimized while $\tilde{f}$ is a maximal M-flow with respect to the new capacities $\tilde{k}(e)$. Here $w(e) \in \mathbb{R}_+$ are the weights (penalties) of changing the capacities $k(e)$ for all $e \in \tilde{E}$. We analyze this problem under rectilinear (ℓ_1) and Chebyshev (ℓ_∞) norms.

4.3.1 Rectilinear (ℓ_1) Norm

Under ℓ_1-norm the objective function of the inverse maximal M-flow problem is

$$\min \sum_{e \in \tilde{E}} w(e)|\tilde{k}(e) - k(e)|. \tag{4.8}$$

In this section, we prove that the inverse maximal M-flow problem on matroid M under ℓ_1-norm can be solved by solving a maximal M-flow problem on another regular matroid M_n. This result generalizes the inverse maximum flow and tension problems under rectilinear norm (Yang *et al.* (1997) and Section 3.1.1).

The given M-flow $\tilde{f}$ is an admissible flow, so it satisfies cocircuit and capacity properties (4.2 and 4.4). However, by max M-flow min cocircuit theorem (Theorem 4.15) we know that there does not exist an f-saturated cocircuit with respect to $\tilde{f}$, because it is not a maximal M-flow. In other words, for all cocircuits $D \in \mathscr{D}_{\tilde{e}}^-$ there exists either

(i) $e \in D^- \backslash \{\tilde{e}\}$ such that $\tilde{f}(e) > 0$, or

(ii) $e \in D^+$ such that $\tilde{f}(e) < k(e)$.

Using this fact we can derive the feasibility condition for inverse maximal M-flow problem.

Lemma 4.18. *If for all cocircuits $D \in \mathscr{D}_{\tilde{e}}^-$ there exists some element $e \in D^- \backslash \{\tilde{e}\}$ such that $\tilde{f}(e) > 0$, then the inverse maximal M-flow problem under ℓ_1-norm is infeasible.*

Proof: Recall that in the inverse problem, we are only allowed to perturb the capacity function k. For all $\tilde{k} \in \mathbb{R}^{|\tilde{E}|}$, the cocircuits $D \in \mathscr{D}_{\tilde{e}}^-$ remain unsaturated. Hence, $\tilde{f}$ would never satisfy the condition of max M-flow min cocircuit theorem (Theorem 4.15) whatever capacity function $\tilde{k}$ we choose.

■

In order to guarantee the feasibility of the inverse problem, let us assume that there exists a $\mathscr{D}^* \subseteq \mathscr{D}_{\tilde{e}}^-$ such that for all $D \in \mathscr{D}^*$, $\tilde{f}(e) = 0$ holds if $e \in D^- \backslash \{\tilde{e}\}$. We can reformulate the inverse maximal M-flow problem as follows.

Lemma 4.19. *The inverse maximal M-flow problem under ℓ_1-norm is equivalent to finding a cocircuit $D \in \mathscr{D}^*$ such that*

$$\sum_{e \in D^+} w(e)(k(e) - \tilde{f}(e)) \tag{4.9}$$

is minimum among all cocircuits in $\mathscr{D}^*$.

Proof: Since all cocircuits in $\mathscr{D}_{\tilde{e}}^-$ are unsaturated, so are the cocircuits in $\mathscr{D}^* \subseteq \mathscr{D}_{\tilde{e}}^-$. Then, by definition of $\mathscr{D}^*$, for all $D \in \mathscr{D}^*$ there exists $e \in D^+$ such that $\tilde{f}(e) < k(e)$. Hence, we have to force one of the cocircuits in $\mathscr{D}^*$ to be saturated by perturbing k such that the condition of max M-flow min cocircuit theorem (Theorem 4.15) will be valid for the given admissible M-flow $\tilde{f}$. The only way to achieve this is to assign $\tilde{k}(e) = \tilde{f}(e)$ for those $e \in D^+$ with $\tilde{f}(e) < k(e)$. Then, the result follows by the definition of the objective function for the inverse problem under ℓ_1-norm.

■

Let us now define a new matroid $M_n = (E_n, \mathscr{C}_n)$ whose ground set contains the dublicated elements of E with positive M-flow, i.e. $E_n := E \cup \bar{F}$ where

$$F := \{e \in \tilde{E} : \tilde{f}(e) > 0\} \quad \text{and} \quad \bar{F} := \{e' : e \in F\}.$$

Moreover, we use the denotations

$$\tilde{E}_n = E_n \backslash \{\tilde{e}\} \quad \text{and} \quad \overline{L \cap F} := \{e' \in \bar{F} : e \in L \cap F\}$$

for the elements of the ground set except the distinguished element $\tilde{e}$ and the set of dublicated elements in M_n corresponding to a given set $L \subseteq E$, respectively. The circuit set of M_n is

$$\mathscr{C}_n := \{\gamma_{F'}(C) : F' \subseteq F,\ C \in \mathscr{C}\} \quad \text{where} \quad \gamma_{F'}(C) = (C \backslash F') \cup \overline{C \cap F'},$$

and the cocircuit set is $\mathscr{D}_n := \{D \cup (\overline{F \cap D}) : D \in \mathscr{D}\}$.

Lemma 4.20. *The matroid $M_n = (E_n, \mathscr{C}_n)$ is a regular matroid with circuit set $\mathscr{C}_n$ and cocircuit set $\mathscr{D}_n$.*

Proof: First of all, we need to show that $M_n = (E_n, \mathscr{C}_n)$ is indeed a matroid. We can achieve this by using Theorem 4.2.

For (C1): Since $\mathscr{C}$ is the circuit set of M, $\emptyset \notin \mathscr{C}$. Hence, by definition of $\mathscr{C}_n$, $\emptyset \in \mathscr{C}_n$ if and only if there exists $F' \subseteq F$ such that $C \backslash F' = \emptyset$ and $C \cap F' = \emptyset$. Since the former condition implies that $C = F'$, both conditions can only be satisfied if $C = \emptyset$, which contradicts the fact that C is a circuit. Thus, $\emptyset \notin \mathscr{C}_n$.

For (C2): Assume that $C_1^n, C_2^n \in \mathscr{C}_n$ and $C_1^n \subseteq C_2^n$ where $C_1^n = \gamma_{F_1}(C_1)$ and $C_2^n = \gamma_{F_2}(C_2)$.

$$\begin{aligned} C_1^n \subseteq C_2^n \quad &\Longrightarrow \quad \text{(i) } C_1 \backslash F_1 \subseteq C_2 \backslash F_2 \quad \text{and} \\ &\qquad\quad \text{(ii) } \overline{C_1 \cap F_1} \subseteq \overline{C_2 \cap F_2} \\ \text{by (ii)} \quad &\Longrightarrow \quad \text{(iii) } C_1 \cap F_1 \subseteq C_2 \cap F_2 \end{aligned}$$

As a result of (i) and (iii), $C_1 \subseteq C_2$, which implies that $C_1 = C_2$ since $\mathscr{C}$ is the circuit set of M. Then, we can conclude from (i) that $F_2 \subseteq F_1$ and from (iii) that $F_1 \subseteq F_2$. Thus, $F_1 = F_2$ and $C_1^n = C_2^n$.

For (C3): Assume again $C_1^n, C_2^n \in \mathscr{C}_n$ and $e \in C_1^n \cap C_2^n$ where $C_1^n = \gamma_{F_1}(C_1)$ and $C_2^n = \gamma_{F_2}(C_2)$. By definition of $\mathscr{C}_n$, either

$$e \in (C_1 \backslash F_1) \cap (C_2 \backslash F_2) \quad \text{or} \quad e \in (\overline{C_1 \cap F_1}) \cap (\overline{C_2 \cap F_2}).$$

This implies that there exists $e' \in (C_1 \cap C_2)$. Since $\mathscr{C}$ satisfies (C3), there exists $C_3 \in \mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \backslash \{e'\}$. Now, we can define $C_3^n := (C_3 \backslash F_3) \cup (\overline{C_3 \cap F_3})$ where $F_3 = F_1 \cup F_2$. Obviously, $C_3^n \subseteq (C_1^n \cup C_2^n) \backslash \{e\}$ and (C3) holds for $\mathscr{C}_n$.

Next, we need to prove that $M_n = (E_n, \mathscr{C}_n)$ is regular, i.e., the circuits and cocircuits of M_n satisfy (4.1). Let us define the partitions to be

$$\begin{aligned} C_n^+ &:= (C^+ \backslash F') \cup (\overline{C^- \cap F'}), \qquad & D_n^+ &:= D^+ \cup (\overline{D^- \cap F}),\\ C_n^- &:= (C^- \backslash F') \cup (\overline{C^+ \cap F'}), \qquad & D_n^- &:= D^- \cup (\overline{D^+ \cap F}), \end{aligned}$$

where $C_n \in \mathscr{C}_n$ and $D_n \in \mathscr{D}_n$ are the circuits and cocircuits of M_n, respectively, and $C \in \mathscr{C}$ and $D \in \mathscr{D}$ are the corresponding circuits and cocircuits of M.

Consider $|C_n^+ \cap D_n^+| + |C_n^- \cap D_n^-|$.

$$\begin{aligned} |C_n^+ \cap D_n^+| &= \left| \Big((C^+ \backslash F') \cup (\overline{C^- \cap F'}) \Big) \cap \Big(D^+ \cup (\overline{D^- \cap F}) \Big) \right| \\ &= \left| \big((C^+ \backslash F') \cap D^+ \big) \cup \Big((\overline{C^- \cap F'}) \cap (\overline{D^- \cap F}) \Big) \right| \\ \text{since } F' \subseteq F \qquad &= \left| \big((C^+ \cap D^+) \backslash F' \big) \cup (\overline{C^- \cap D^- \cap F'}) \right| \\ &= |(C^+ \cap D^+) \backslash F'| + |C^- \cap D^- \cap F'| \end{aligned}$$

Analogously, we can show $|C_n^- \cap D_n^-| = |(C^- \cap D^-) \backslash F'| + |C^+ \cap D^+ \cap F'|$. As a result,

$$\begin{aligned} |C_n^+ \cap D_n^+| + |C_n^- \cap D_n^-| &= |(C^+ \cap D^+) \backslash F'| + |C^- \cap D^- \cap F'| \\ &\quad + |(C^- \cap D^-) \backslash F'| + |C^+ \cap D^+ \cap F'| \\ &= |C^+ \cap D^+| + |C^- \cap D^-|. \end{aligned} \tag{4.10}$$

Furthermore,

$$|C_n^+ \cap D_n^-| + |C_n^- \cap D_n^+| = |C^+ \cap D^-| + |C^- \cap D^+|. \tag{4.11}$$

Since M is a regular matroid, the circuits C and cocircuits D satisfy (4.1). Then, by (4.10) and (4.11), M_n satisfies (4.1) and is a regular matroid.

∎

Now we are ready to prove our main result of inverse maximal M-flow problems under weighted ℓ_1-norm.

Theorem 4.21. *Consider a maximal M-flow problem on $M_n = (E_n, \mathscr{C}_n)$ with a capacity function $k_n : \tilde{E}_n \to \mathbb{R}^{|\tilde{E}_n|}$ where*

$$k_n(e) = \begin{cases} w(e)(k(e) - \tilde{f}(e)) & \forall\ e \in \tilde{E} \\ \infty & \textit{otherwise} \end{cases}$$

and $r_n(e) = 0$ for all $e \in \tilde{E}_n$. If this maximal M-flow problem is unbounded, then the inverse maximal M-flow problem on M with respect to the admissible M-flow $\tilde{f}$ is infeasible under the weighted ℓ_1-norm. If the optimum objective function value of the maximal M-flow problem on M_n is K^* with a minimum cocircuit D_{min}, then the optimum objective function value of the inverse problem is also K^*. Moreover,

$$k^*(e) = \begin{cases} \tilde{f}(e) & \forall\ e \in D^+_{min} \\ k(e) & \textit{otherwise} \end{cases}$$

is an optimum solution to the inverse maximal M-flow problem on M under the weighted ℓ_1-norm.

Proof: First, suppose that the maximal M-flow problem on M_n is unbounded, which means by max M-flow min cocircuit theorem (Theorem 4.15) that for all cocircuits $D_n \in \mathscr{D}_n$ with $\tilde{e} \in D_n^-$ there exists an element $e \in D_n^+$ with an infinite capacity. By definition of the capacities k_n this implies that $(\overline{D^- \cap F}) \neq \emptyset$ for all $D \in \mathscr{D}_{\tilde{e}}{}^-$, i.e, there exists an element $e \in D^- \backslash \{\tilde{e}\}$ with $\tilde{f}(e) > 0$. Thus, by Lemma 4.18, the inverse problem is infeasible.

If the maximal M-flow problem on M_n is bounded with a maximal M-flow of K^*, then there exists a minimum cocircuit $D_{min} \in \mathscr{D}_n$ with

$$K^* = \sum_{e \in D^+_{min}} k_n(e) = \sum_{e \in D^+} w(e)(k(e) - \tilde{f}(e))$$

which is by Lemma 4.19 an optimum solution to the inverse problem on M.

■

This result can be easily extended to the case $r(e) \neq 0$ where we can perturb both k and r. In order to achieve this we need to redefine $F := \{e \in \tilde{E} : \tilde{f}(e) > r(e)\}$. Obviously, in this case the inverse problem is always feasible.

Corollary 4.22. *Consider a maximal M-flow problem on $M_n = (E_n, \mathscr{C}_n)$ with a capacity* *function* $k_n : \tilde{E}_n \to \mathbb{R}$ *where*

$$k_n(e) = \begin{cases} w(e)(k(e) - \tilde{f}(e)) & \forall\, e \in \tilde{E} \\ w(e)(\tilde{f}(e) - r(e)) & \forall\, e \in \tilde{E}_n \backslash \tilde{E} \end{cases}$$

and $r_n(e) = 0$ *for all* $e \in \tilde{E}_n$*. If the optimum objective function value of the maximal M-flow problem on* M_n *is* K^* *with a minimum cocircuit* D_{min}*, then the optimum objective function value of inverse problem is also* K^**. Moreover,*

$$k^*(e) = \begin{cases} \tilde{f}(e) & \forall\, e \in D^+_{min} \cap \tilde{E} \\ k(e) & \textit{otherwise} \end{cases} \qquad r^*(e) = \begin{cases} \tilde{f}(e) & \forall\, e \in \tilde{E} \;\textit{ with }\; e' \in D^+_{min} \cap \bar{F} \\ r(e) & \textit{otherwise} \end{cases}$$

is an optimum solution to the inverse maximal M-flow problem on M *under* ℓ_1*-norm.*

4.3.2 Chebyshev (ℓ_∞) Norm

Under ℓ_∞-norm the objective function of the inverse maximal M-flow problem is

$$\min \max_{e \in \tilde{E}} w(e)|\tilde{k}(e) - k(e)|. \tag{4.12}$$

Analogous to the rectilinear case, we can show that the feasibility result of Lemma 4.18 is valid for the Chebyshev case and modify Lemma 4.19 for the new norm. Let us define $\mathscr{D}^* \subseteq \mathscr{D}^-_{\tilde{e}}$ such that for all $D \in \mathscr{D}^*$, $\tilde{f}(e) = 0$ holds if $e \in D^- \backslash \{\tilde{e}\}$, as previously.

Lemma 4.23. *The inverse maximal M-flow problem under* ℓ_∞*-norm is finding a cocircuit* $D \in \mathscr{D}^*$ *such that*

$$\max_{e \in D^+} w(e)(k(e) - \tilde{f}(e)) \tag{4.13}$$

is minimum among all cocircuits in $\mathscr{D}^*$*.*

Recall the definition of a blocking system and their corresponding property (Property 2.5) from Section 2.1. We can use this definition and property to prove the following result:

Corollary 4.24. *If there exists an* f*-augmenting circuit* $C \in \mathscr{C}^+_{\tilde{e}}$ *with respect to* $\tilde{f}$ *such that* $C^+ \backslash \{\tilde{e}\} = \emptyset$*, then the inverse maximal M-flow problem under* ℓ_∞*-norm is infeasible. The inverse maximal M-flow problem under* ℓ_∞*-norm is finding an* f*-augmenting circuit* $C \in \mathscr{C}^+_{\tilde{e}}$ *with respect to* $\tilde{f}$ *such that* $C^+ \backslash \{\tilde{e}\} \neq \emptyset$ *and*

$$\min_{e \in C^+ \backslash \{\tilde{e}\}} w(e)(k(e) - \tilde{f}(e)) \tag{4.14}$$

is maximum among all f*-augmenting circuits in* $\mathscr{C}^+_{\tilde{e}}$*.*

Proof: Let us first prove the feasibility condition. By definition of the inverse maximal M-flow problem, we are only allowed to change $k(e)$ for $e \in \tilde{E}$ such that $\tilde{f}$ will be a maximal M-flow. If there exists an f-augmenting circuit $C \in \mathscr{C}^+_{\tilde{e}}$ with respect to $\tilde{f}$ such that $C^+ \backslash \{\tilde{e}\} = \emptyset$, no matter how we perturb the capacities $k(e)$, the circuit C stays to be an f-augmenting circuit. Hence, the inverse problem is infeasible.

Now, let us assume that there does not exist an f-augmenting circuit $C \in \mathscr{C}^+_{\tilde{e}}$ with respect to $\tilde{f}$ such that $C^+ \backslash \{\tilde{e}\} = \emptyset$. Define $\mathfrak{R}$ to be the sets of elements in D^+ for all cocircuits $D \in \mathscr{D}^*$ and $\mathfrak{S}$ to be the sets of elements in $C^+ \backslash \{\tilde{e}\}$ for all f-augmenting circuits with respect to $\tilde{f}$. We need to show the validity of Property 2.5 for $\mathfrak{R}$ and $\mathfrak{S}$.

Consider the capacity function $k : \tilde{E} \to \{0, 1\}$. We define

$$E_0 = \{e \in E : k(e) = 0\} \quad \text{and} \quad E_1 = \{e \in E : k(e) = 1\}.$$

If the maximal M-flow is equal to 0, then there exists a cocircuit D with $D^+ \subseteq E_0$. But, there does not exist a circuit C, which is augmenting with respect to 0 M-flow, i.e., there does not exist $C = C^+$ with $C^+ \backslash \{\tilde{e}\} \subseteq E_1$. Similarly if the maximal M-flow equals 1, by max M-flow min Cocircuit theorem (Theorem 4.15) there exists a cocircuit D with an element $e \in D^+$ having $k(e) = 1$. In other words, there does not exist $D^+ \in \mathfrak{R}$ such that $D^+ \subseteq E_0$. On the other hand, there exists an f-augmenting circuit C with respect to 0 M-flow, i.e., $C = C^+$ and $C^+ \backslash \{\tilde{e}\} \subseteq E_1$. Consequently, $\mathfrak{R}$ and $\mathfrak{S}$ form a blocking system by Property 2.5, and the result follows.

■

Next, we will show that the inverse maximal M-flow problem under ℓ_∞-norm can be solved by identifying a circuit $C_f \in \mathscr{C}^+_{f,\tilde{e}}$ with $C_f = C_f^+$ among the circuits $\mathscr{C}_f$ of the incremental matroid, which maximizes

$$\min_{e \in C_f \backslash \{\tilde{e}\}} k_f(e), \tag{4.15}$$

where $k_f : \tilde{E}_f \to \mathbb{R}$ is the incremental capacity.

Let us first provide the definition of the *incremental matroid* M_f with respect to $\tilde{f}$. Following Burkard and Hamacher (1981) we denote $F_1 := \{e \in \tilde{E} : \tilde{f}(e) < k(e)\} \cup \{\tilde{e}\}$ and $F_2 := \{e \in \tilde{E} : \tilde{f}(e) > r(e)\}$. Each element $e \in F := F_1 \cap F_2$ is dublicated and denoted by e'. $\bar{F} := \{e' : e \in F\}$ is the set of all dublicated elements. Then, the ground set of the incremental matroid M_f is $E_f := E \cup \bar{F}$ and $\tilde{E}_f$ denote the ground set without the distinguished element, i.e., $\tilde{E}_f := E_f \backslash \{\tilde{e}\}$. Moreover, if we use the definitions of $\overline{L \cap F}$ and $\gamma_{F'}(C)$ as in Section 4.3.1,

$$\mathscr{C}_f := \{\gamma_{F'}(C) : F' \subseteq F,\ C \in \mathscr{C}\} \quad \text{and} \quad \mathscr{D}_f := \{D \cup (\overline{F \cap D}) : D \in \mathscr{D}\}$$

define the circuit and cocircuit sets of the incremental matroid, respectively. Notice that in our case $r(e) = 0$ for all $e \in \tilde{E}$ of the regular matroid M.

Lemma 4.25. *(Burkard and Hamacher, 1981) There exists a circuit C in M with $\tilde{f}(e) < k(e)$ for all $e \in C^+$ and $\tilde{f}(e) > r(e)$ for all $e \in C^-$ if and only if there exists a circuit $C_f \in \mathscr{C}_f$ with $C_f^- = \emptyset$.*

We define the capacity function $k_f : \tilde{E}_f \to \mathbb{R}$ for the incremental matroid M_f with respect to $\tilde{f}$ as

$$k_f(e) := \begin{cases} w(e)(k(e) - \tilde{f}(e)) & \text{if} \quad e \in F_1 \backslash \{\tilde{e}\} \\ K & \text{otherwise} \end{cases} \tag{4.16}$$

where K is a sufficiently large real number.

By Lemma 4.25, we know that there exists an f-augmenting circuit in $\mathscr{C}$ with respect to $\tilde{f}$ if there exists a circuit C_f of incremental matroid M_f with $C_f^- = \emptyset$ and $\tilde{e} \in C_f$. Moreover, if $\max \min_{e \in C_f \backslash \{\tilde{e}\}} k_f(e) = K$, then there exists an f-augmenting circuit $C \in \mathscr{C}_{\tilde{e}}^+$ with respect to $\tilde{f}$ such that $C^+ \backslash \{\tilde{e}\} = \emptyset$ and by Corollary 4.24 the inverse problem is infeasible. Otherwise, we can identify an f-augmenting circuit $C \in \mathscr{C}_{\tilde{e}}^+$ of matroid M satisfying Corollary 4.24.

For the special cases of flows and tensions on directed graphs, we have shown that the corresponding inverse problems can be solved as maximum capacity path problems (see sections 2.1 and 3.1.2).

4.4 Cost Inverse Minimum Cost M-Flow Problem

In this section we consider the cost inverse problem of minimum cost M-flows. We are given an instance of a minimum cost M-flow problem on a regular matroid M with capacity functions $k, r : \tilde{E} \to \mathbb{R}^{|\tilde{E}|}$ and cost function $a : E \to \mathbb{R}^{|E|}$ for which $a(\tilde{e}) = 0$. We also have a maximal M-flow $\tilde{f}$ which is not a minimum cost M-flow. The *cost inverse mininum cost M-flow problem* is, then, perturbing the costs $a(e)$ to $\tilde{a}(e)$ for $e \in \tilde{E}$ such that

$$\|\tilde{a} - a\| \tag{4.17}$$

is minimized while $\tilde{f}$ is a minimum cost M-flow with respect to the new costs $\tilde{a}$.

Before we look into this inverse problem under rectilinear (ℓ_1) and Chebyshev (ℓ_∞) norms, we analyze a new optimality condition for the minimum cost M-flows. This optimality condition indeed generalizes the ***kilter optimality conditions*** and was already mentioned by Burkard and Hamacher (1981) without proof. We review this result and provide our own proof for the sake of the completeness of the further results. Recall that if a maximal M-flow is satisfying the conditions (4.18), then it is *in-kilter*, otherwise it is *out-of-kilter*.

Consider the dual matroid of M, which we denote by $M_d = (E, \mathscr{D})$. By definition of dual matroids, the circuits of M are the cocircuits of M_d and M_d is also a regular matroid.

Theorem 4.26. *Given a maximal M-flow f on the regular matroid $M = (E, \mathscr{C})$. f is a minimum cost M-flow if and only if there exists an M-flow γ on the dual matroid $M_d = (E, \mathscr{D})$ satisfying for all $e \in \tilde{E}$,*

$$\begin{aligned} &\gamma(e) \leq a(e) \quad \textit{if} \quad f(e) = r(e) \\ &\gamma(e) = a(e) \quad \textit{if} \quad r(e) < f(e) < k(e) \\ &\gamma(e) \geq a(e) \quad \textit{if} \quad f(e) = k(e). \end{aligned} \tag{4.18}$$

Proof: "$\Leftarrow$" Suppose that there exists an M-flow γ on M_d satisfying (4.18), but f is not a minimum cost M-flow. Since γ is an M-flow on M_d, it satisfies the cocircuit property, i.e.,

$$\gamma(C^+) = \sum_{e \in C^+} \gamma(e) = \sum_{e \in C^-} \gamma(e) = \gamma(C^-) \quad \forall\, C \in \mathscr{C}.$$

Moreover, since f is not a minimum cost M-flow by Theorem 4.17, there exists a negative circuit $C \in \mathscr{C} \setminus \mathscr{C}_{\tilde{e}}^-$ in M. Consider the cocircuit property of γ for this negative circuit C,

$$\begin{aligned} \gamma(C^+) - \gamma(C^-) &= \sum_{e \in C^+} \gamma(e) - \sum_{e \in C^-} \gamma(e) \\ &\leq \sum_{e \in C^+} a(e) - \sum_{e \in C^-} a(e) \\ &< 0. \end{aligned}$$

Here, the first inequality comes from the fact that γ satisfies the conditions (4.18) and the second inequality from the fact that C is a negative circuit in M. Clearly, this result contradicts the initial assumption that γ is an M-flow on M_d.

"$\Rightarrow$" Suppose that f is a minimum cost M-flow but there does not exist an M-flow γ on M_d satisfying (4.18). There exist two violations of (4.18) to consider:

(V1) Suppose that there exists $e^* \in \tilde{E}$ with $f(e^*) < k(e^*)$ and $\gamma(e^*) > a(e^*)$. Without loss of generality (wlog), we can assume that $e^* \in C^+$ for some $C \in \mathscr{C}$ (otherwise we can replace C^+ by C^- using the fact that M is a regular matroid).

Recall that C is a cocircuit for the dual matroid M_d, hence, γ fulfills the cocircuit property on C, i.e.

$$\gamma(C) = \sum_{e \in C^+} \gamma(e) - \sum_{e \in C^-} \gamma(e) = 0.$$

Our aim here is, using γ, to construct a new M-flow γ_{new} on M_d that satisfies (4.18). In order to achieve this, we assign $\gamma_{new}(e^*) := a(e^*) < \gamma(e^*)$. This disturbs the cocircuit property on C, forcing $\gamma(C)$ to be negative. Therefore, we have to set either $\gamma_{new}(e) > \gamma(e)$ for some $e \in C^+$ or $\gamma_{new}(e) < \gamma(e)$ for some

$e \in C^-$. This can be achieved easily without violating the conditions (4.18) if there exists

- an $e \in C^+$ with $f(e) = k(e)$ by assigning $\gamma_{new}(e) > \gamma(e) \geq a(e)$,
- or an $e \in C^-$ with $f(e) = r(e)$ by assigning $\gamma_{new}(e) < \gamma(e) \leq a(e)$.

Thus, we will assume that $\forall\, e \in C^+,\ f(e) < k(e)$ and $\forall\, e \in C^-,\ f(e) > r(e)$.

- $\tilde{e} \notin C$ **:** Since all $e \in C\backslash\{e^*\}$ satisfy (4.18), we can write

$$\sum_{e\in C^+} \gamma(e) - \sum_{e\in C^-} \gamma(e) \leq \sum_{e\in C^+} a(e) - \sum_{e\in C^-} a(e) + (\gamma(e^*) - a(e^*)).$$

  Since f is a minimum cost M-flow, there does not exist any negative circuits, i.e.

$$\sum_{e\in C^+} a(e) - \sum_{e\in C^-} a(e) \geq 0.$$

  by using this fact and the initial assumption on $\gamma(e^*) - a(e^*) > 0$ we can rearrange the above inequality as

$$\sum_{e\in C^+} (\gamma(e) - a(e)) - \sum_{e\in C^-} (\gamma(e) - a(e)) < \gamma(e^*) - a(e^*),$$

  which implies that there exists

  (i) either an $e \in C^+\backslash\{e^*\}$ such that $\gamma(e) < a(e)$,

  (ii) or an $e \in C^-$ such that $\gamma(e) > a(e)$.

  Therefore, we can find a new M-flow γ_{new} on M_d with $\gamma(e) < \gamma_{new}(e) \leq a(e)$ for (i), $\gamma(e) > \gamma_{new}(e) \geq a(e)$ for (ii), and $\gamma(e^*) > \gamma_{new}(e^*) := a(e^*)$ and this new M-flow on M_d satisfies both cocircuit property and the conditions of (4.18).

- $\tilde{e} \in C^+$**:** In this case the circuit C would define an f-augmenting circuit, which contradicts to the maximality of the flow f on regular matroid M. Hence, such a circuit $C \in \mathscr{C}$ cannot exist.

- $\tilde{e} \in C^-$**:** In this case we can reduce the value of M-flow on $\tilde{e}$ until the cocircuit property is fulfilled, i.e., assign $\gamma_{new}(\tilde{e}) < \gamma(\tilde{e})$.

(V2) Suppose that there exists $e^* \in \tilde{E}$ with $f(e^*) > r(e^*)$ and $\gamma(e^*) < a(e^*)$. Similar to **(V1)**, we can assume wlog that $e^* \in C^-$ for some $C \in \mathscr{C}$. Analogous to **(V1)** we assign $\gamma_{new}(e^*) := a(e^*) > \gamma(e^*)$, which disturbs the cocircuit property on C, forcing $\gamma(C)$ to be negative. In order to reestablish the cocircuit property, we have to set either $\gamma_{new}(e) > \gamma(e)$ for some $e \in C^+$ or $\gamma_{new}(e) < \gamma(e)$ for some $e \in C^-$. Hence, the discussions of **(V1)** are valid for this case, as well.

As it can be seen, we can eliminate the violations of the conditions (4.18) on a given circuit $C \in \mathscr{C}$ by constructing a new M-flow γ_{new} on M_d. By repetitive application of the above arguments, we can generate an admissible M-flow on M_d fulfilling (4.18) if the given M-flow f on M is a minimum cost M-flow. Thus, the result of the theorem follows.

■

4.4.1 Rectilinear (ℓ_1) Norm

Under unit weight rectilinear norm our aim is to perturb the costs $a(e)$ to $\tilde{a}(e)$ for $e \in \tilde{E}$ such that

$$\sum_{e \in \tilde{E}} |\tilde{a}(e) - a(e)| \tag{4.19}$$

is minimized while $\tilde{f}$ is a minimum cost M-flow with respect to the new costs $\tilde{a}$. In this section, we generalize the results on the inverse minimum cost flow and tension problems and prove that the inverse minimum cost M-flow problem under unit weight ℓ_1-norm can be solved by finding a minimum cost collection of disjoint residual circuits on matroid M with respect to $\tilde{f}$. For this purpose, we need to define *disjoint residual circuits*. This definition is actually analogous to the definitions of arc-disjoint residual cycles and cuts.

Definition 4.27. We call two residual circuits C_1 and C_2 to be *disjoint* if $C_1^+ \cap C_2^+ = \emptyset$ and $C_1^- \cap C_2^- = \emptyset$.

Now similar to the flow and tension cases we show for the M-flows that the following property holds.

Property 4.28. *Let $\mathscr{C}_R^* = \{C_1^*, C_2^*, \ldots, C_K^*\}$ denote a minimum cost collection of disjoint residual circuits with respect to a given maximal (but not minimum cost) M-flow $\tilde{f}$ on matroid M. Then, there exists an M-flow γ on the dual matroid M_d such that*

$$\text{if } e \in \mathscr{C}_R^* \quad : \quad a(e) - \gamma(e) \begin{cases} \leq 0 & \text{for } e \in \mathscr{C}_R^{*+} \\ \geq 0 & \text{for } e \in \mathscr{C}_R^{*-} \end{cases} \tag{4.20a}$$

$$\text{if } e \notin \mathscr{C}_R^* \quad : \quad a(e) - \gamma(e) \begin{cases} \geq 0 & \text{for } \tilde{f}(e) < k(e) \\ \leq 0 & \text{for } \tilde{f}(e) > r(e) \end{cases} \tag{4.20b}$$

Proof: The proof of this property is quite similar to the proof of Theorem 4.26. As in the proof of Theorem 4.26 we assume initially that there exists a violation and then show that this violation is avoidable by changing the dual M-flow values. Notice that since $\mathscr{C}_R^*$ is a minimum cost collection of disjoint residual circuits and $\tilde{f}$ is not a minimum cost M-flow, all the residual circuits in $\mathscr{C}_R^*$ are negative circuits.

Let us assume that there exists $e^* \in C_k^{*+}$ for $C_k^* \in \mathscr{C}_R^*$ such that $a(e^*) - \gamma(e^*) > 0$ and e^* is the only element on C_k^* violating (4.20a). Then,

$$\sum_{e \in C_k^{*+}} a(e) - \sum_{e \in C_k^{*-}} a(e) < \left(\sum_{e \in C_k^{*+}} \gamma(e) - \sum_{e \in C_k^{*-}} \gamma(e) \right) + a(e^*) - \gamma(e^*), \tag{4.21}$$

holds since γ is an M-flow and satisfies the cocircuit property, i.e.,

$$\sum_{e \in C_k^{*+}} \gamma(e) - \sum_{e \in C_k^{*-}} \gamma(e) = 0,$$

and C_k^* is a negative circuit. This implies that there exists

(i) either an $e' \in C_k^{*+}$ such that $a(e') < \gamma(e')$,

(ii) or an $e' \in C_k^{*-}$ such that $a(e') > \gamma(e')$.

By using e' we can find a new M-flow γ_{new} such that $\gamma_{new}(e^*) := a(e^*)$ and $a(e') \leq \gamma_{new}(e') < \gamma(e')$ for (i) or $\gamma(e') < \gamma_{new}(e') \leq a(e')$ for (ii). By applying this alteration iteratively we can obtain an M-flow satisfying (4.20a).

Suppose that the violation occurs for an element $e^* \notin \mathscr{C}_R^*$. Consider a circuit for which $e^* \in C$ and $e \notin \mathscr{C}_R^*$ holds for all $e \in C$. If C is not a residual circuit, then there exists $e' \in C$ with $\tilde{f}(e') = k(e')$ or $\tilde{f}(e') = r(e')$, or $\tilde{e} \in C$. Then, we can use e' or $\tilde{e}$ to generate γ_{new} which avoids the violation but fulfills the cocircuit property. Let us further assume that C is a residual circuit, wlog $\tilde{f}(e^*) < k(e^*)$ and $a(e^*) - \gamma(e^*) < 0$. By assumption, C is a disjoint circuit with $\mathscr{C}_R^*$ but $C \notin \mathscr{C}_R^*$. Therefore, it has a nonnegative cost. Moreover,

$$\sum_{e \in C^+} a(e) - \sum_{e \in C^-} a(e) > \sum_{e \in C^+} \gamma(e) - \sum_{e \in C^-} \gamma(e) + (a(e^*) - \gamma(e^*)),$$

since all $e \in C \backslash \{e^*\}$ satisfy (4.20b) and C is a residual circuit. This again implies that there exists

- either an $e' \in C^+$ such that $a(e') > \gamma(e')$,
- or an $e' \in C^-$ such that $a(e') < \gamma(e')$.

Consequently, the previous discussions are valid for this case, as well.

∎

After we have shown the validity of Property 4.28, the extension of Theorem 3.13 over to the M-flows on regular matroids is pretty straightforward. Therefore, we only provide a summarized proof for the following theorem.

Theorem 4.29. *Suppose* $\mathscr{C}^*_R = \{C^*_1, C^*_2, \ldots, C^*_K\}$ *denotes a minimum cost collection of disjoint residual circuits on* M *with respect to a given maximal M-flow* $\tilde{f}$*, which is not a minimum cost M-flow. Let* $Cost(\mathscr{C}^*_R)$ *be the cost of* $\mathscr{C}^*_R$*, which is equal to the total costs of the negative circuits in* $\mathscr{C}^*_R$*. Then,* $-Cost(\mathscr{C}^*_R)$ *is the optimal objective function value for the inverse minimum cost M-flow problem under unit weight rectilinear norm.*

Proof: The proof follows quite analogous to the proofs of the special cases flows and tensions (see Theorem 3.13). Suppose that $\mathscr{C}^*_R = \{C^*_1, C^*_2, \ldots, C^*_K\}$ denotes any collection of disjoint residual circuits on M with respect to $\tilde{f}$ and let a^* denote the optimum cost vector for the inverse problem. First of all, by using similar arguments as in proof of Theorem 3.13 we can show that $-Cost(\mathscr{C}^*_R)$ is a lower bound on $\|a^* - a\|_1$. Then, we can prove that this lower bound is actually achievable for a minimum cost collection of disjoint residual circuits by setting $a^*(e) = \gamma(e)$ for $e \in \mathscr{C}^*_R$ and $a^*(e) = a(e)$ otherwise, where γ is the dual M-flow satisfying Property 4.28. Clearly, this cost vector is in-kilter (by Theorem 4.26), hence it is a feasible solution to the inverse problem. Moreover,

$$\begin{aligned}
\|a^* - a\|_1 &= \sum_{e \in \mathscr{C}^{*-}_R} (a(e) - \gamma(e)) - \sum_{e \in \mathscr{C}^{*+}_R} (a(e) - \gamma(e)) \\
&= -\sum_{k=1}^{K} \left(\sum_{e \in C^{*+}} a(e) - \sum_{e \in C^{*-}} a(e) \right) \qquad (4.22) \\
&= -Cost(\mathscr{C}^*_R)
\end{aligned}$$

Thus, the result of the theorem follows.

■

Next, we will show that a minimum cost collection of disjoint residual circuits can be found by solving a minimum cost M-flow problem on the *incremental matroid* $M_f = (E_f, \mathscr{C}_f)$ of M with respect to M-flow $\tilde{f}$. For this purpose, recall the definition of incremental matroid given in Section 4.3.2 and the statement of Lemma 4.25.

Theorem 4.30. *Consider a minimum cost M-flow problem on the incremental matroid* M_f *with respect to* $\tilde{f}$ *with costs*

$$a_f(e) := \begin{cases} a(e) & \text{if } e \in F_1, \\ -a(e) & \text{if } e \in F_2 \backslash F, \\ -a(e') & \text{if } e \in \bar{F} \text{ and } e' \in F, \end{cases} \qquad (4.23)$$

and bounds $r(e) = 0$ *and* $k(e) = 1$ *for all* $e \in \tilde{E}_f$*. The cost of a minimum cost M-flow* f *with the flow value* $f(\tilde{e}) = 0$ *is equal to the total cost of a minimum cost collection of disjoint residual circuits on* M *with respect to* $\tilde{f}$*. Moreover, one of the* ***positive circuit decompositions*** *of the M-flow f on M_f provides a minimum cost collection of disjoint residual circuits on M with respect to $\tilde{f}$.*

Before we prove the theorem, we need to define a *positive circuit decomposition* of an M-flow f. For a detailed analysis of the decomposition of M-flows on regular matroids, we refer to Hamacher (1982a).

Recall the definition of an M-circulation (Definition 4.12) and the relationship to the M-flows (Theorem 4.13). A *circuit decomposition* of an M-flow f is an M-circulation g such that $f \equiv f_g$. If it is a positive circuit decomposition, then the resulting M-circulation only attains values $g(C) > 0$ for circuits with $C = C^+$. Hamacher (1982a) proved that for an M-flow with $f(e) \geq 0$ for all $e \in E$, there always exists a positive circuit decomposition.

Proof of Theorem 4.30: Let f be a minimum cost M-flow on the incremental matroid M_f with respect to $\tilde{f}$ and $f(\tilde{e}) = 0$. By Equivalence Theorem (4.13), there exists an M-circulation g with $f \equiv f_g$, which is a positive circuit decomposition for f. Moreover, $f_g(\tilde{e}) = 0$, which implies by (4.3) that for all circuits C_f with $C_f = C_f^+$ and $\tilde{e} \in C_f$, $g(C_f) = 0$. Then, by Lemma 4.25, we know that only the circuits $C_f \in \mathscr{C}_f{}^+$ of M_f corresponding to the residual circuits of M with respect to $\tilde{f}$ satisfy $0 < g(C_f) \leq 1$.

Now consider the objective function of the minimum M-flow problem on M_f.

$$\begin{aligned} \sum_{e \in E_f} f(e) a_f(e) = \sum_{e \in E_f} f_g(e) a_f(e) &= \sum_{e \in E_f} \left(\sum_{C_f \in \mathscr{C}_{f,e}^+} g(C_f) \right) a_f(e) \\ &= \sum_{C_f \in \mathscr{C}_f^+} g(C_f) \left(\sum_{e \in C_f} a_f(e) \right) \end{aligned} \tag{4.24}$$

By definition of the costs a_f on the incremental matroid M_f

$$\sum_{e \in C_f} a_f(e) = \sum_{e \in C^+} a(e) - \sum_{e \in C^-} a(e)$$

for all circuits $C_f \in \mathscr{C}_f{}^+$, where C is a corresponding residual circuit of M by Lemma 4.25. In objective function (4.24), the minimum can be achieved if $g(C_f) = 1$ for all circuits C_f corresponding the elements of a minimum cost collection of disjoint residual circuits of M with respect to $\tilde{f}$. Since the residual circuits of M are disjoint, so are the circuits C_f with $g(C_f) = 1$. Thus, for all $e \in E_f$, $0 \leq f(e) \leq 1$ is satisfied and the result of the theorem follows.

■

By Theorem 4.30, we have shown that the cost inverse minimum cost M-flow problem on a regular matroid M under unit weight rectilinear norm is equivalent to solving a minimum cost M-flow problem on the incremental matroid with unit flow capacities.

4.4.2 Chebyshev (ℓ_∞) Norm

Under Chebyshev norm our aim is to perturb the costs $a(e)$ to $\tilde{a}(e)$ for all $e \in \tilde{E}$ such that

$$\max_{e \in \tilde{E}} |\tilde{a}(e) - a(e)| \tag{4.25}$$

is minimized while $\tilde{f}$ is a minimum cost M-flow with respect to the new costs $\tilde{a}$. In this section, we generalize the results on inverse minimum cost flow and tension problems and prove that inverse minimum cost M-flow problem under unit weight Chebyshev norm can be solved by finding a minimum mean cost residual circuit on matroid M with respect to $\tilde{f}$. In order to achieve this, we have to first carry over the notion of ϵ-optimality to M-flows on regular matroids.

Definition 4.31. For an $\epsilon \geq 0$, an M-flow f on the regular matroid $M = (E, \mathscr{C})$ is ***ϵ-optimal*** if there exists an M-flow γ on the dual matroid $M_d = (E, \mathscr{D})$ such that

$$\forall\, e \in \tilde{E} : \begin{cases} (f(e) < k(e)) & \Longrightarrow (\gamma(e) \leq a(e) + \epsilon) \\ (f(e) > r(e)) & \Longrightarrow (\gamma(e) \geq a(e) - \epsilon) \end{cases} \tag{4.26}$$

Theorem 4.32. *A maximal M-flow f on the regular matroid $M = (E, \mathscr{C})$ is ϵ-optimal if and only if every residual circuit C has a mean cost of*

$$MCost(C) = \frac{a(C)}{|C|} \geq -\epsilon.$$

Proof: **"$\Rightarrow$"** Suppose that the M-flow f on the regular matroid $M = (E, \mathscr{C})$ is ϵ-optimal but there exists a residual circuit $C \in \mathscr{C}_R$ with a mean cost of $MCost(C) < -\epsilon$. Since C is a cocircuit for the dual matroid M_d, the M-flow γ fulfills the cocircuit property on C, i.e.,

$$\sum_{e \in C^+} \gamma(e) - \sum_{e \in C^-} \gamma(e) = 0.$$

By definition of ϵ-optimality (4.31) we can show that

$$\sum_{e \in C^+} \gamma(e) - \sum_{e \in C^-} \gamma(e) \leq \sum_{e \in C^+} (a(e) + \epsilon) - \sum_{e \in C^-} (a(e) - \epsilon).$$

If we reorder the inequality and use the cocircuit property, we get

$$\sum_{e \in C^+} a(e) - \sum_{e \in C^-} a(e) + |C|\epsilon \geq 0,$$

which implies that $MCost(C) \geq -\epsilon$. This is a contradiction to the initial assumption.

"$\Leftarrow$" Suppose that every residual circuit $C \in \mathscr{C}_R$ has $MCost(C) \geq -\epsilon$. Then,

$$\sum_{e \in C^+} a(e) - \sum_{e \in C^-} a(e) \geq -\epsilon |C|, \quad \text{i.e.,}$$

$$\sum_{e \in C^+} (a(e) + \epsilon) - \sum_{e \in C^-} (a(e) - \epsilon) \geq 0.$$

This implies that all $C \in \mathscr{C}_R$ will have nonnegative costs if we assign a new cost function a^* such that

$$a^*(e) = \begin{cases} a(e) + \epsilon & \text{if} \quad e \in C^+ \quad \text{for } C \in \mathscr{C}_R, \\ a(e) - \epsilon & \text{if} \quad e \in C^- \quad \text{for } C \in \mathscr{C}_R, \\ a(e) & \text{if} \quad e \in C_1^+ \cap C_2^- \quad \text{for } C_1, C_2 \in \mathscr{C}_R, \\ a(e) & \text{otherwise.} \end{cases}$$

Consequently, the M-flow f is a minimum cost M-flow on matroid M with respect to the cost function a^* by Negative Circuit Theorem (4.17). Then, by Theorem 4.26, there exists an M-flow on M_d satisfying (4.18), i.e., there exists γ such that

$$\gamma(e) \begin{cases} \leq a^*(e) = a(e) + \epsilon & \text{if} \quad f(e) < k(e) \\ \geq a^*(e) = a(e) - \epsilon & \text{if} \quad f(e) > r(e). \end{cases}$$

Thus, by Definition 4.31, f is ϵ-optimal.

■

The definition of ϵ-optimality (4.26) and Theorem 4.32 imply the following property of the M-flows.

Property 4.33. *Let C_m be a minimum mean residual circuit on a regular matroid M with respect to a given maximal M-flow $\tilde{f}$, which is not a minimum cost M-flow. Let μ^* be the mean cost of it. There exists a dual M-flow γ on M_d such that $a(e) - \gamma(e) = \mu^*$ for $e \in C_m^+$ and $a(e) - \gamma(e) = -\mu^*$ for $e \in C_m^-$. All other elements $e \in \tilde{E}$ satisfy $a(e) - \gamma(e) \geq \mu^*$ if $\tilde{f}(e) < k(e)$ and $a(e) - \gamma(e) \leq -\mu^*$ if $\tilde{f}(e) > r(e)$.*

Now, using Property 4.33, we can prove the following theorem, which is the main result of this section.

Theorem 4.34. *Let μ^* denote the mean cost of a minimum mean (cost) residual circuit on a regular matroid M with respect to a given maximal M-flow $\tilde{f}$, which is not a minimum cost M-flow. Then, the optimal objective function value for the inverse minimum cost M-flow problem under unit weight ℓ_∞-norm is $max(0, -\mu^*)$.*

Proof: The proof of this theorem also uses analogous arguments as the proofs of special cases, e.g. of Theorem 3.19. We choose γ as in Property 4.33. Let z^* be the optimum solution to the inverse minimum cost M-flow problem under unit weight Chebyshev norm. We can first show that $z^* \geq -\mu^*$ by using similar arguments as in the proof of Theorem 3.19. Moreover, if we define the new cost vector a^* as

$$a^*(e) = \begin{cases} a(e) - \mu^* & \text{if } \tilde{f}(e) < k(e) \text{ and } a(e) < \gamma(e) \\ a(e) + \mu^* & \text{if } \tilde{f}(e) > r(e) \text{ and } a(e) > \gamma(e) \\ a(e) & \text{otherwise,} \end{cases} \tag{4.27}$$

then $\tilde{f}$ satisfies the optimality conditions of Theorem 4.26 and a^* is an optimal solution of the inverse minimum cost M-flow problem under unit weight Chebyshev norm since $\|a^* - a\|_\infty \leq -\mu^*$.

∎

Theorem 4.35. *A minimum mean residual circuit on a regular matroid M with respect to $\tilde{f}$ can be found by solving a minimum cost M-flow problem on the incremental matroid M_f with respect to $\tilde{f}$ such that the M-flow $f \geq 0$ on M_f satisfies*

$$f(\tilde{e}) = 0 \quad \textit{and} \quad \sum_{e \in E_f} f(e) = 1. \tag{4.28}$$

Proof: Let f be a minimum cost M-flow on M_f satisfying the given conditions (4.28). Consider a positive circuit decomposition of the M-flow f. We claim that a positive circuit decomposition of f contains only minimum mean cost residual circuits of matroid M with respect to $\tilde{f}$.

Suppose that the claim is not true. First of all, we know that $f_g(\tilde{e}) = 0$ since, by assumption, M-flow f on M_f fulfills the conditions (4.28). This implies by (4.3) that for all circuits C_f with $C_f = C_f^+$ and $\tilde{e} \in C_f$, $g(C_f) = 0$. Then, by Lemma 4.25, only the circuits $C_f \in \mathscr{C}_f{}^+$ of M_f associated with the residual circuits of M with respect to $\tilde{f}$ satisfy $g(C_f) > 0$. Hence, a positive circuit decomposition of f consists only of those circuits C_f, which correspond to the residual circuits of M.

Furthermore, each circuit C_f with M-circulation $g(C_f) > 0$ contributes

$$\left(g(C_f) \sum_{e \in C_f} a_f(e) \right)$$

to the objective function and $g(C_f)|C_f|$ to the condition that the sum of the total M-flow values is equal to 1.

Now suppose that there exists a positive circuit decomposition with 2 circuits C_f^1 and C_f^2 on M_f, but C_f^2 is not a minimum mean circuit. Then,

$$\frac{\sum_{e \in C_f^1} a_f(e)}{|C_f^1|} < \frac{\sum_{e \in C_f^2} a_f(e)}{|C_f^2|}. \tag{4.29}$$

Moreover, by the condition (4.28)

$$\sum_{e \in E_f} f(e) = 1 \quad \Rightarrow \quad \sum_{e \in C_f^1} f(e) + \sum_{e \in C_f^2} f(e) = 1$$
$$g(C_f^1)|C_f^1| + g(C_f^2)|C_f^2| = 1. \tag{4.30}$$

Let us consider another feasible flow f^* on M_f, which contains only C_f^1 in a positive circuit decomposition, i.e., $f^* = g^*$. In this case, $g^*(C_f^1)|C_f^1| = 1$ and the objective function value is

$$\frac{\sum_{e \in C_f^1} a_f(e)}{|C_f^1|}. \tag{4.31}$$

Since M-flow f is a minimum cost M-flow, the objective function value with respect to f must be as good as the objective function value for f^*, i.e.

$$g(C_f^1) \sum_{e \in C_f^1} a_f(e) + g(C_f^2) \sum_{e \in C_f^2} a_f(e) \leq g^*(C_f^1) \sum_{e \in C_f^1} a_f(e).$$

$$\begin{aligned}
g(C_f^1) \sum_{e \in C_f^1} a_f(e) + g(C_f^2) \sum_{e \in C_f^2} a_f(e) &= g(C_f^1) \sum_{e \in C_f^1} a_f(e) + \left(\frac{1 - g(C_f^1)|C_f^1|}{|C_f^2|} \right) \sum_{e \in C_f^2} a_f(e) \\
\text{(by (4.30))} \quad &= g(C_f^1) \sum_{e \in C_f^1} a_f(e) + \frac{\sum_{e \in C_f^2} a_f(e)}{|C_f^2|} \left(1 - g(C_f^1)|C_f^1|\right) \\
\text{(by (4.29))} \quad &> g(C_f^1) \sum_{e \in C_f^1} a_f(e) + \frac{\sum_{e \in C_f^1} a_f(e)}{|C_f^1|} \left(1 - g(C_f^1)|C_f^1|\right) \\
&= \frac{\sum_{e \in C_f^1} a_f(e)}{|C_f^1|}
\end{aligned}$$

which is the objective function value of f^* by (4.31). Consequently, we get a contradiction and the initial claim is true.

■

Notice that the given M-flow $\tilde{f}$ in Theorem 4.34 is a maximal M-flow. In this case, we do not need to have the condition $f(\tilde{e}) = 0$ in Theorem 4.35. If $\tilde{f}$ is a maximal M-flow, then there does not exist any circuits of the incremental matroid such that $C_f = C_f^+$ with $\tilde{e} \in C_f$. Otherwise, such a circuit C_f corresponds to an f-augmenting circuit in M, which contradicts to the assumption that $\tilde{f}$ is a maximal M-flow. A similar argument is also valid for Theorem 4.30 of Section 4.4.1.

4.5 Capacity Inverse Minimum Cost M-Flow Problem

In this section we consider the capacity inverse problem of minimum cost M-flows under Chebyshev norm and prove that the greedy algorithm solves this problem, as well.

4.5.1 Problem Definition

We are given an instance of a minimum cost M-flow problem on a regular matroid M with capacity functions $k, r : \tilde{E} \to \mathbb{R}^{|\tilde{E}|}$ and cost function $a : E \to \mathbb{R}^{|E|}$ for which $a(\tilde{e}) = 0$. We also have a maximal M-flow $\tilde{f}$ which is not a minimum cost M-flow. The *capacity inverse mininum cost M-flow problem* is, then, perturbing the capacity functions from $k(e)$ and $r(e)$ to $\tilde{k}(e)$ and $\tilde{r}(e)$ for $e \in \tilde{E}$ such that

$$\|(\tilde{k}, \tilde{r}) - (k, r)\| \tag{4.32}$$

is minimized while $\tilde{f}$ is a minimum cost M-flow with respect to the new capacity functions $\tilde{k}$ and $\tilde{r}$.

Since the given M-flow $\tilde{f}$ is a nonoptimal maximal M-flow, by Negative Circuit Theorem (Theorem 4.17), there exists negative circuits with respect to $\tilde{f}$. In order to solve the capacity inverse problem, we have to modify the parameters k and r to $\tilde{k}$ and $\tilde{r}$ in a way that there does not exist any negative circuits with respect to $\tilde{f}$ under the new capacity parameters $(\tilde{k}, \tilde{r})$.

Let us consider the incremental matroid $M_f = (\tilde{E}_f, \mathscr{C}_f)$ with respect to $\tilde{f}$ with a cost function defined by (4.23), a capacity function $k_f : \tilde{E}_f \to \mathbb{R}_+$ such that

$$k_f(e) := \begin{cases} k(e) - \tilde{f}(e) & \text{if} \quad e \in F_1 \backslash \{\tilde{e}\}, \\ \tilde{f}(e) - r(e) & \text{if} \quad e \in F_2 \backslash F, \\ \tilde{f}(e') - r(e') & \text{if} \quad e \in \bar{F} \text{ and } e' \in F, \end{cases} \tag{4.33}$$

and $r_f(e) = 0$ for all $e \in \tilde{E}_f$. Using the incremental matroid M_f we can reformulate the capacity inverse minimum cost M-flow problem as follows:

Proposition 4.36. *The capacity inverse minimum cost M-flow problem is equivalent to finding a subset S of $\tilde{E}_f$ such that S includes at least one element from each negative cost circuit C_f of M_f with $C_f = C_f^+$. Moreover, among all the subsets of $\tilde{E}_f$ satisfying this condition, we would like to find the one minimizing $\|k_f\|$ according to a given norm.*

Proof: First of all recall that there does not exist a circuit C_f of the incremental matroid such that $C_f = C_f^+$ and $\tilde{e} \notin C_f$. If such a circuit exists, then it defines an f-augmenting circuit with respect to $\tilde{f}$, which is a contradiction to the maximality of the given M-flow $\tilde{f}$.

By Lemma 4.25 we know that for each negative circuit C of M with respect to $\tilde{f}$ there exists a circuit $C_f = C_f^+$ in incremental matroid with negative cost. As mentioned previously, we have to modify the parameters $k(e)$ and $r(e)$ of the elements $e \in C$ for the negative circuits so that we can eliminate the negative circuits and the given M-flow satisfies the optimality condition. This can be achieved either by setting $\tilde{k}(e) := \tilde{f}(e)$ for some $e \in C^+$ or by assigning $\tilde{r}(e) := \tilde{f}(e)$ for some $e \in C^-$ for each negative circuit $C \in \mathscr{C}$. The effect of this modification on the incremental matroid is that for one of the elements $e \in \tilde{E}_f$ the incremental capacity is modified to $k_f(e) = 0$. Indeed, the definition of the incremental matroid changes since the sets F_1, F_2 and $\bar{F}$ are modified. Hence, the result follows.

■

4.5.2 Chebyshev (ℓ_∞) Norm

The capacity inverse mininum cost M-flow problem under ℓ_∞-norm is perturbing the capacity functions from $k(e)$ and $r(e)$ to $\tilde{k}(e)$ and $\tilde{r}(e)$ for $e \in \tilde{E}$ such that

$$\max_{e \in \tilde{E}} \{\max\{|\tilde{k}(e) - k(e)|, |\tilde{r}(e) - r(e)|\}\} \tag{4.34}$$

is minimized while $\tilde{f}$ is a minimum cost M-flow with respect to the new capacity functions $\tilde{k}$ and $\tilde{r}$. We show in this section that the following greedy algorithm solves this problem optimally, analogous to the special cases of matroid flows.

Algorithm 11. (Greedy Algorithm for Matroid Flows)

1. Initialize the incremental matroid $M_f = (E_f, \mathscr{C}_f)$ and the sets of affected elements $S_k = S_r = \emptyset$ with $S_k, S_r \subseteq \tilde{E}$.

2. Choose a negative cost circuit C_f with $C_f = C_f^+$ from the input incremental matroid $M_f = (E_f, \mathscr{C}_f)$.
   IF there exists no negative cost circuits STOP.
   Output: For $e \notin S_k$ assign $\tilde{k}(e) = k(e)$, else set $\tilde{k}(e) = \tilde{f}(e)$ and for $e \notin S_r$ assign $\tilde{r}(e) = r(e)$, else set $\tilde{r}(e) = \tilde{f}(e)$ with objective value
   $$\max\{\max_{e \in S_k} k(e) - \tilde{f}(e), \max_{e \in S_r} \tilde{f}(e) - r(e)\}.$$

3. Find $e_m = \arg\min_{e \in C_f} k_f(e)$, then set $S_k := S_k \cup \{e_m\}$ if $e_m \in F_1 \backslash \{\tilde{e}\}$ or set $S_r := S_r \cup \{e_m\}$ if $e_m \in F_2 \backslash F$ or $S_r := S_r \cup \{e'_m\}$ if $e_m \in \bar{F}$.
   Update the incremental matroid. GO TO Step-2.

Theorem 4.37. *The greedy algorithm solves the capacity inverse minimum cost M-flow problem under unit weight ℓ_∞-norm optimally.*

Proof: Obviously, the algorithm delivers a feasible solution to the problem since it stops only when there does not exist any negative circuits. In order to show the optimality, let us assume that (k^*, r^*) is an optimal solution of the capacity inverse minimum cost M-flow problem under unit weight ℓ_∞-norm and $(k^*, r^*) \neq (\tilde{k}, \tilde{r})$ where $(\tilde{k}, \tilde{r})$ is the solution delivered by greedy algorithm. Then,

$$\max_{e \in \tilde{E}} \{\max\{|k^*(e) - k(e)|, |r^*(e) - r(e)|\}\} \leq \max_{e \in \tilde{E}} \{\max\{|\tilde{k}(e) - k(e)|, |\tilde{r}(e) - r(e)|\}\}.$$

By construction of the greedy algorithm and Proposition 4.36, there exists a negative circuit C^* such that

$$\begin{aligned} &\arg\max_{e \in \tilde{E}} \{\max\{|\tilde{k}(e) - k(e)|, |\tilde{r}(e) - r(e)|\}\} =: e^* \in C^* \\ \text{and} \qquad & e^* = \arg\min\{\min_{e \in C^{*+}} (k(e) - \tilde{f}(e)), \min_{e \in C^{*-}} (\tilde{f}(e) - r(e))\} \end{aligned}$$

Then, for all $e \in C^*$,

$$\begin{aligned} \max_{e \in \tilde{E}} \{\max\{|k^*(e) - k(e)|, |r^*(e) - r(e)|\}\} &\leq \max_{e \in \tilde{E}} \{\max\{|\tilde{k}(e) - k(e)|, |\tilde{r}(e) - r(e)|\}\} \\ &\leq \min\{\min_{e \in C^{*+}} (k(e) - \tilde{f}(e)), \min_{e \in C^{*-}} (\tilde{f}(e) - r(e))\} \end{aligned}$$

By Proposition 4.36, we know that at least one element from each negative circuit must have a capacity value assigned to the value of M-flow on that element, i.e., either $k^*(e) = \tilde{f}(e)$ or $r^*(e) = \tilde{f}(e)$ for $e \in C$ where C is a negative circuit with respect to $\tilde{f}$. Therefore, there exists $e' \in C^*$ such that

$$\begin{aligned} \max\{(k^*(e') - \tilde{f}(e')), (\tilde{f}(e') - r^*(e'))\} &\leq \max_{e \in \tilde{E}} \{\max\{|k^*(e) - k(e)|, |r^*(e) - r(e)|\}\} \\ &\leq \max_{e \in \tilde{E}} \{\max\{|\tilde{k}(e) - k(e)|, |\tilde{r}(e) - r(e)|\}\} \\ &\leq \min\{\min_{e \in C^{*+}} (k(e) - \tilde{f}(e)), \min_{e \in C^{*-}} (\tilde{f}(e) - r(e))\} \\ &\leq \max\{(\tilde{k}(e') - \tilde{f}(e')), (\tilde{f}(e') - \tilde{r}(e'))\} \end{aligned}$$

Here all the inequalities must hold with equality since (k^*, r^*) is an optimal solution. Thus, $(k^*, r^*) = (\tilde{k}, \tilde{r})$.

∎

In this thesis we do not provide any generalized algorithm to find a negative circuit with respect to a given M-flow, because the generalizations of the flow and tension algorithms to matroid flows is beyond the scope of this thesis. Consequently, we do not analyze the time complexity of the greedy algorithm for matroid flows, either. Such an analysis is left for future research.

The shortest path between two truths in the real domain passes through the complex domain.

J.S. Hadamard (1865-1963)

5 Cost Inverse Problems of Monotropic Programs

Monotropic programming deals with optimization problems that minimize a separable convex function subject to linear constraints. Several optimization problems such as linear and piecewise linear programs, quadratic and piecewise quadratic programs, network flows and tensions are special cases of monotropic programs. In this chapter, we analyze inverse problems of monotropic programs with separable linear cost functions and show that the combinatorial solutions of Ahuja and Orlin (2002) can be extended to these problems. Section 5.1 introduces the theory of monotropic programming. In Section 5.2 we analyze the inverse primal problem with separable linear costs under ℓ_1-norm, whereas Section 5.3 focuses on the same problem under ℓ_∞-norm. In Section 5.4 we analyze the generalized minimum cost flow problem as a special case of monotropic programs, which do not possess totally unimodularity.

5.1 Introduction to Monotropic Optimization

The theory of monotropic programming was first established by Rockafellar (1984) and extended mainly by Bertsekas (1998); Tseng and Bertsekas (1990, 2000); Tseng (2001). Here we provide a brief introduction to monotropic programming and refer to the book of Rockafellar (1984) for details.

Monotropic programming deals with optimization problems that minimize a separable convex function subject to linear constraints written in the following form

$$\text{Minimize} \quad \Phi(x) = \sum_{j \in J} f_j(x_j) \tag{P}$$

subject to

$$y_i = \sum_{j \in J} e(i,j) x_j = b_i \qquad \forall\, i \in I$$

$$x_j \in C_j \qquad \forall\, j \in J.$$

Here, $E = e(i,j)$ is an arbitrary real matrix expressed in terms of nonempty and finite index sets I and J. Each $f_j : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a closed, proper convex function and C_j is the interval where f_j is finite. We call (P) the *primal problem*.

We denote the left and right derivatives of f_j at ξ with $f_j^-(\xi)$ and $f_j^+(\xi)$, respectively, and extend these functions from $C_j = [c_j^-, c_j^+]$ to $\mathbb{R}$ by defining

$$\begin{array}{llll} f_j^-(\xi) = f_j^+(\xi) = +\infty & \text{if } \xi > c_j^+ \quad \text{and} & f_j^+(\xi) = +\infty & \text{if } \xi = c_j^+, \\ f_j^-(\xi) = f_j^+(\xi) = -\infty & \text{if } \xi < c_j^- \quad \text{and} & f_j^-(\xi) = -\infty & \text{if } \xi = c_j^-. \end{array}$$

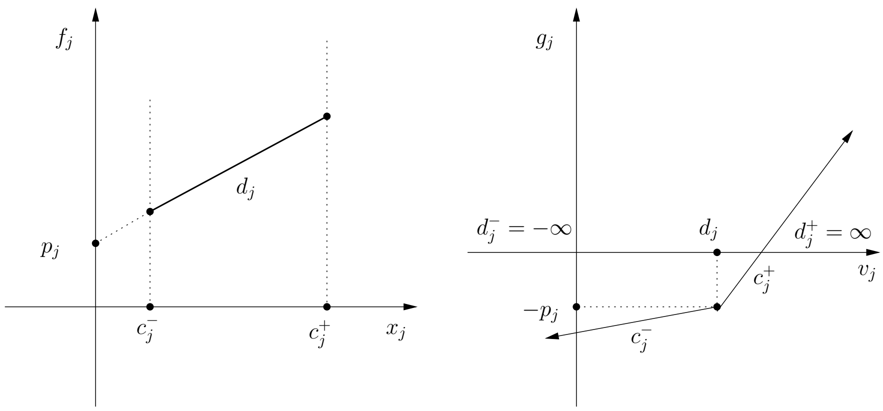

Figure 5.1: Example of conjugate cost functions

The *dual problem* of (P) is of the form

$$\text{Maximize} \quad \Psi(u) = -\sum_{i \in I} b_i u_i - \sum_{j \in J} g_j(v_j) \tag{D}$$

subject to

$$v_j = -\sum_{i \in I} u_i e(i,j) \qquad \forall\, j \in J$$

$$v_j \in D_j \qquad \forall\, j \in J$$

where g_j denotes the conjugate function of f_j, i.e.,

$$g_j(v_j) = \sup_{\xi \in \mathbb{R}} \{v_j \xi - f_j(\xi)\}$$

and D_j is the interval in which g_j is finite (see Figure 5.1). By definition, the respective subspaces of the primal and dual problems,

$$\mathcal{C} = \{x : Ex = 0\}$$
$$\mathcal{D} = \{v : \exists\, u \,\text{with } -uE = v\},$$

are orthogonally complementary to each other. Graphically, this means that (x_j, v_j) is on the *characteristic curve* Γ_j (see Figure 5.2), i.e., $(x_j, v_j) \in \Gamma_j$ where

$$\Gamma_j = \{(\xi, \eta) \in \mathbb{R}^2 : f_j^-(\xi) \leq \eta \leq f_j^+(\xi)\} \qquad \forall\, j \in J.$$

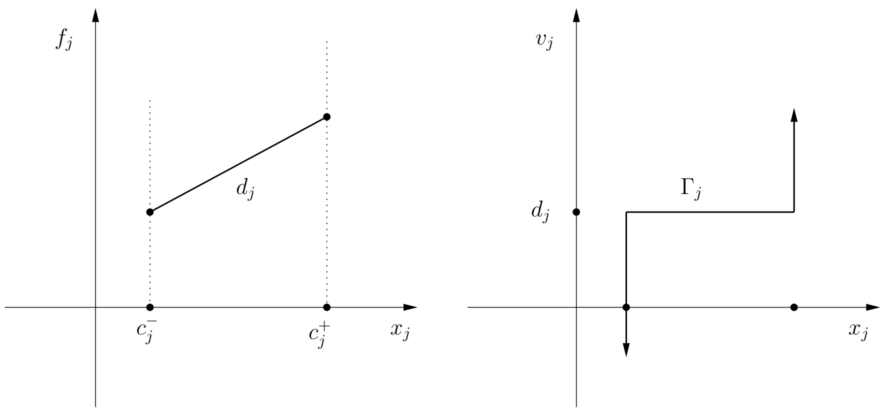

Figure 5.2: A linear cost function (5.1) and the corresponding characteristic curve

In this study, we will assume that there exists a feasible solution x to the primal problem (P), satisfying

$$f_j^-(x_j) < \infty \quad \text{and} \quad f_j^+(x_j) > -\infty \qquad \forall\, j \in J.$$

Such an x is called *regularly feasible solution* of (P). Moreover, we will consider only the special case where the cost function of (P) is separable linear, i.e.,

$$f_j(x_j) = \begin{cases} d_j x_j & \text{if} \quad c_j^- \leq x_j \leq c_j^+, \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}$$

5.2 Inverse Primal Problem with Linear Costs under ℓ_1 Norm

In the inverse problem of (P), we are given a regularly feasible solution $\tilde{x}$, which is not optimal. Our aim is to modify the cost functions f_j such that the given solution $\tilde{x}$ will be optimum for the new cost functions while the perturbation of the cost is minimized according to some norm. Under the rectilinear norm, we would like to perturb d_j to $\tilde{d}_j$ for which $\tilde{x}$ is an optimum solution to (P) and $\sum_{j\in J}|\tilde{d}_j - d_j|$ is minimum.

First of all, we repeat some of the basic definitions and results on the optimality of monotropic programs where we refer again to Rockafellar (1984) for details.

Definition 5.1. A signed subset P of J is called a ***support*** of $\mathcal{C}$, or a *primal support*, if there is a vector $x \in \mathcal{C}$ such that

$$P^+ = \{j \in J : x_j > 0\} \quad \text{and} \quad P^- = \{j \in J : x_j < 0\}.$$

A primal support P is *elementary* if it is nonempty and does not properly include any other primal support. For an elementary support P, we define an ***elementary vector*** e_P to be the unique elementary $x \in \mathcal{C}$ having P as its support and satisfying

$$|e_P(j)| \leq 1. \tag{5.2}$$

Hence,

$$\sum_{j\in P^+} e_P(j) - \sum_{j\in P^-} e_P(j) \leq |P|. \tag{5.3}$$

Note that this definition of the elementary vector e_P is different from the definition given in Rockafellar (1984) where he normalizes $x \in \mathcal{C}$ to get e_P such that the inequality (5.3) holds with equality. However, in this section we normalize $x \in \mathcal{C}$ to get e_P such that (5.2) holds. This new normalization of elementary primal support vector is necessary for the future discussions.

Definition 5.2. An elementary primal support P gives an ***elementary direction of descent at*** $\tilde{x}$ if and only if

$$Cost(P) = \sum_{j\in P^+} f_j^+(\tilde{x}_j)e_P(j) + \sum_{j\in P^-} f_j^-(\tilde{x}_j)e_P(j) < 0. \tag{5.4}$$

Theorem 5.3. *A feasible solution $\tilde{x}$ to the primal problem is optimal if and only if there is no elementary direction of descent for Φ at $\tilde{x}$ (Rockafellar, 1984).*

In the primal problem, the given regularly feasible solution $\tilde{x}$ is not optimum. Hence, there exists at least one elementary direction of descent for Φ at $\tilde{x}$. By using the definition of f_j (5.1) and the existence conditions of left and right derivatives of

f_j, we can conclude that there exists an elementary vector e_P such that

$$\text{for } j \in P^+ \Rightarrow \tilde{x}_j < c_j^+ \quad \text{and} \quad \text{for } j \in P^- \Rightarrow \tilde{x}_j > c_j^- \tag{5.5}$$

$$\text{and } Cost(P) = \sum_{j \in P} d_j e_P(j) < 0. \tag{5.6}$$

Since the aim of the inverse problem is to change the cost vector d until there does not exist any elementary supports defining a descent direction at $\tilde{x}$ while minimizing the total change, the following lemma is easy to justify.

Lemma 5.4. *There exists an optimal solution $\tilde{d}$ of the inverse problem for which $\tilde{d}_j \geq d_j$ for all $j \in P^+$ and $\tilde{d}_j \leq d_j$ for all $j \in P^-$ where P is an elementary support defining a descent direction at the given solution $\tilde{x}$.*

Following the denotations of previous sections, we call two elementary primal supports P_1 and P_2 *disjoint* if $P_1^+ \cap P_2^+ = \emptyset$ and $P_1^- \cap P_2^- = \emptyset$.

Rockafellar (1984) mentions that solving the primal optimality problem is equivalent to solving the dual feasibility problem with respect to the dual spans $D_x(j) = [d_x^-(j), d_x^+(j)]$ where

$$d_x^+(j) = f_j^+(\tilde{x}_j) = \begin{cases} \infty & \text{if } \tilde{x}_j = c_j^+ \\ d_j & \text{if } \tilde{x}_j < c_j^+ \end{cases}, \tag{5.7}$$

$$d_x^-(j) = f_j^-(\tilde{x}_j) = \begin{cases} d_j & \text{if } \tilde{x}_j > c_j^- \\ -\infty & \text{if } \tilde{x}_j = c_j^- \end{cases}$$

for the linear cost function $f_j(x_j)$ defined by (5.1). According to our assumption $\tilde{x}$ is a feasible nonoptimal solution. Hence, the dual problem with respect to the spans $D_x(j)$ for $j \in J$ is infeasible and the following property holds.

Property 5.5. *Let $\mathcal{P} = \{P_1, \ldots, P_K\}$ be a minimum cost collection of disjoint elementary primal supports defining descent direction at $\tilde{x}$. For the elementary primal supports in $\mathcal{P}$ there exists a $v \in \mathcal{D}$ such that*

$$\textit{For } j \in \mathcal{P} \begin{cases} v_j \geq d_j & \textit{if } j \in \mathcal{P}^+ \\ v_j \leq d_j & \textit{if } j \in \mathcal{P}^- \end{cases}, \textit{ and for } j \notin \mathcal{P} \begin{cases} v_j \leq d_j & \textit{if } \tilde{x}_j < c_j^+ \\ v_j \geq d_j & \textit{if } \tilde{x}_j > c_j^- \end{cases} \tag{5.8}$$

Moreover, we can find a v for which the inequalities (5.8) hold and $v_j = d_j$ for all $|e_P(j)| \neq 1$.

Proof: In order to prove that the inequalities (5.8) hold we use a constructive approach similar to the proof of Property 4.28 for matroid flows in regular matroids. We assume again initially that there exists a violation and then show that this violation is avoidable by changing the dual vector v. Suppose wlog that there exists a support

$P \in \mathcal{P}$ such that there exists $k \in P^+$ and $v_k < d_k$. Then,

$$\sum_{j\in P^+} d_j e_P(j) + \sum_{j\in P^-} d_j e_P(j) \leq \left(\sum_{j\in P^+} v_j e_P(j) + \sum_{j\in P^-} v_j e_P(j) \right) + (d_k - v_k) e_P(k).$$

By definition of v we know that $v.e_P = 0$ and the elementary supports in $\mathcal{P}$ define descent directions at $\tilde{x}$. Thus, the inequality is a strict inequality, which implies that

- either $\exists\, j \in P^+$ such that $d_j < v_j$,
- or $\exists\, j \in P^-$ such that $d_j > v_j$.

This means that we can iteratively construct a new dual vector v satisfying the inequalities (5.8).

Next, we will show that the last claim is true, i.e., there exists $v \in \mathcal{D}$ for which the inequalities (5.8) hold and $v_j = d_j$ for all $|e_P(j)| \neq 1$. Suppose that there exists a $P_k \in \mathcal{P}$ for which the claim does not hold, i.e., there does not exist $v \in \mathcal{D}$ such that for $j \in P_k$ with $|e_{P_k}(j)| \neq 1$ the equality $v_j = d_j$ holds. Assume wlog that $0 < e_{P_k}(j) < 1$. Since $v \in \mathcal{D}$ and $v_l = d_l$ for all $l \in \{t \in P_k : |e_{P_k}(t)| \neq 1\} \backslash \{j\}$,

$$\sum_{l\in\{t\in P_k:|e_{P_k}(t)|\neq 1\}} e_{P_k}(l) d_l + \sum_{l\in\{t\in P_k:|e_{P_k}(t)|=1\}} e_{P_k}(l) v_l + e_{P_k}(j) v_j = 0$$

and $v_j > d_j$ by the inequalities (5.8). Recall that by the normalization of the elementary vector e_P (5.2) there exists at least one $j \in P_k$ for which $|e_{P_k}(j)| = 1$.

Suppose we set $v_j = d_j$. As the elementary primal supports are disjoint, the effect of this change will only be on P_k. In order to establish the balance, we need to increase either v_l for $l \in \{j \in P_k : e_{P_k}(j) = 1\}$ or decrease v_l for $l \in \{j \in P_k : e_{P_k}(j) = -1\}$. In either case the new v satisfies $v \in \mathcal{D}$ and the Property 5.5 holds with $v_j = d_j$ for all $|e_P(j)| \neq 1$. Hence, the claim is true.

■

Now using Property 5.5 we can prove the following theorem, which extends the results of inverse network flows and tensions under rectilinear norm to the inverse monotropic programs with separable linear cost functions.

Theorem 5.6. *Let $\mathcal{P} = \{P_1, \ldots, P_K\}$ be a minimum cost collection of disjoint elementary primal supports defining descent direction at $\tilde{x}$. The objective function value of the inverse primal problem under unit weight ℓ_1-norm is $-Cost(\mathcal{P}) = -\sum_{k=1}^{K} Cost(P_k)$.*

Proof: First of all, we will show that $-Cost(\mathcal{P})$ is a lower bound on the objective function value. By definition of e_P, we know that

$$\text{for } j \in P^+ \begin{cases} e_P(j)d_j \leq d_j & \text{if } d_j \geq 0 \\ e_P(j)d_j \geq d_j & \text{if } d_j \leq 0 \end{cases} \tag{5.9a}$$

$$\text{for } j \in P^- \begin{cases} e_P(j)d_j \geq -d_j & \text{if } d_j \geq 0 \\ e_P(j)d_j \leq -d_j & \text{if } d_j \leq 0 \end{cases} \tag{5.9b}$$

Then,

$$\begin{aligned}
\sum_{j\in J} |\tilde{d}_j - d_j| \quad &\geq \quad \sum_{k=1}^{K} \sum_{j\in P_k} |\tilde{d}_j - d_j| \\
&= \quad \sum_{k=1}^{K} \left(\sum_{j\in P_k^+} (\tilde{d}_j - d_j)(+1) + \sum_{j\in P_k^-} (\tilde{d}_j - d_j)(-1) \right) \\
(i) \qquad &\geq \quad \sum_{k=1}^{K} \left(\sum_{j\in P_k^+} (\tilde{d}_j - d_j)(e_P(j)) + \sum_{j\in P_k^-} (\tilde{d}_j - d_j)(e_P(j)) \right) \\
&\geq \quad \sum_{k=1}^{K} -Cost(P_k) = -Cost(\mathcal{P})
\end{aligned}$$

Here, the first inequality holds because the elementary primal supports are disjoint and the inequality (i) holds by (5.9a, 5.9b).

In order to complete the proof, we need to show that this lower bound is indeed achievable. We exploit Property 5.5 to define our new cost function $\tilde{d}$. Let $v \in \mathcal{D}$ satisfy Property 5.5. We set $\tilde{d}_j = v_j$ for all $j \in \mathcal{P}$ and $\tilde{d}_j = d_j$ otherwise. Then,

$$\begin{aligned}
\|\tilde{d} - d\|_1 &= \sum_{j\in J} |\tilde{d}_j - d_j| = \sum_{j\in\mathcal{P}} |v_j - d_j| \\
&= -\left(\sum_{k=1}^{K} \sum_{j\in P_k^+} (d_j - v_j)(+1) + \sum_{j\in P_k^-} (d_j - v_j)(-1) \right) \\
(*) \qquad &= -\left(\sum_{k=1}^{K} \sum_{j\in P_k^+} (d_j - v_j)(e_{P_k}(j)) + \sum_{j\in P_k^-} (d_j - v_j)(e_{P_k}(j)) \right) \\
&= -Cost(\mathcal{P})
\end{aligned}$$

Here, the equality $(*)$ holds since $v_j = d_j$ holds for $j \in \mathcal{P}$ and $|e_P(j)| \neq 1$ as shown previously. Hence, the proof of the theorem is complete.

■

5.3 Inverse Primal Problem with Linear Costs under ℓ_∞ Norm

Under Chebyshev norm, we would like to perturb d_j to $\tilde{d}_j$ for which $\tilde{x}$ is an optimum solution to (P) and $\max_{j\in J}|\tilde{d}_j - d_j|$ is minimum.

Following Tseng and Bertsekas (1990), we say that an $x \in \mathbb{R}^{|J|}$ and a $v \in \mathbb{R}^{|J|}$ satisfy ***ϵ-complementary slackness***, where ϵ is any positive scalar, if

$$f_j(x_j) < \infty \quad \text{and} \quad f_j^-(x_j) - \epsilon \;\le\; v_j \;\le\; f_j^+(x_j) + \epsilon, \quad \text{for } j \in J. \tag{5.10}$$

Graphically, this means that (x_j, v_j) is within ϵ vertical distance of the characteristic curve Γ_j (see Figure 5.3). We call x an *ϵ-optimal solution* if x satisfies the ϵ-complementary slackness conditions (5.10).

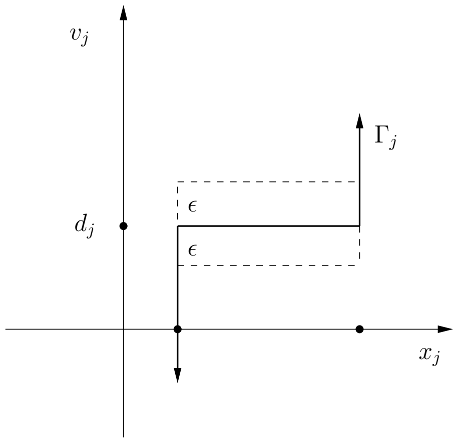

Figure 5.3: The set of points within ϵ vertical distance of Γ_j

Recall the definition of an elementary primal support. For an elementary support P, we define an ***elementary vector*** e_P to be the unique elementary $x \in \mathcal{C}$ having P as its support and satisfying

$$\sum_{j\in P^+} e_P(j) - \sum_{j\in P^-} e_P(j) = |P|. \tag{5.11}$$

Note that this definition of the elementary vector e_P is exactly the definition given in Rockafellar (1984). In the previous section (Section 5.2), we employed another normalization to define the elementary vector.

Theorem 5.7. *A given feasible solution $\tilde{x}$ to (P) is an ϵ-optimal solution if and only if all the elementary primal supports defining a descent direction with respect to $\tilde{x}$ have a mean cost ($MCost(P)$), which is greater than $-\epsilon$, i.e.,*

$$MCost(P) = \frac{Cost(P)}{|P|} = \frac{\sum_{j\in P^+} f_j^+(\tilde{x}_j)e_P(j) + \sum_{j\in P^-} f_j^-(\tilde{x}_j)e_P(j)}{|P|} \ge -\epsilon,$$

where e_P satisfies (5.11).

Proof: "⇒" Suppose that the given feasible solution $\tilde{x}$ is an ϵ-optimal solution. Since $e_P v = 0$ for $v \in \mathcal{D}$, the following holds for any elementary primal support P.

$$\begin{array}{rcl} Cost(P) & = & Cost(P) - e_P v \\ & = & \sum_{j \in P^+} f_j^+(\tilde{x}_j) e_P(j) + \sum_{j \in P^-} f_j^-(\tilde{x}_j) e_P(j) - \sum_{j \in P} e_P(j) v_j \\ & = & \sum_{j \in P^+} (f_j^+(\tilde{x}_j) - v_j) e_P(j) + \sum_{j \in P^-} (f_j^-(\tilde{x}_j) - v_j) e_P(j) \\ (i) & \geq & \sum_{j \in P^+} (-\epsilon) e_P(j) + \sum_{j \in P^-} (\epsilon) e_P(j) \\ & & = -\epsilon \left(\sum_{j \in P^+} e_P(j) - \sum_{j \in P^-} e_P(j) \right) \\ (ii) & = & -\epsilon |P| \end{array}$$

Here, the inequality (i) holds by ϵ-complementary slackness (5.10) whereas (ii) holds by the definition of the elementary vector e_P and (5.11). Hence, it can be concluded that $MCost(P) \geq -\epsilon$ for all elementary primal supports P.

"⇐" Suppose that all the elementary primal supports have $MCost(P) \geq -\epsilon$ but the solution $\tilde{x}$ is not ϵ-optimal,i.e., $f_j^-(x_j) - v_j > \epsilon$ and $f_j^+(x_j) - v_j < -\epsilon$ for $j \in J$. Then,

$$\begin{aligned} Cost(P) &= \sum_{j \in P^+} f_j^+(\tilde{x}_j) e_P(j) + \sum_{j \in P^-} f_j^-(\tilde{x}_j) e_P(j) \\ &< \sum_{j \in P^+} (-\epsilon + v_j) e_P(j) + \sum_{j \in P^-} (\epsilon + v_j) e_P(j) \\ &= \sum_{j \in P^+} (-\epsilon) e_P(j) + \sum_{j \in P^-} (\epsilon) e_P(j) \\ &= -\epsilon \left(\sum_{j \in P^+} e_P(j) - \sum_{j \in P^-} e_P(j) \right) \\ &= -\epsilon |P| \end{aligned}$$

which is a contradiction.

■

Theorem 5.8. *Let P^* be a minimum mean cost elementary primal support defining a descent direction at $\tilde{x}$ and μ^* be its mean cost. The optimum objective function value of inverse primal problem with linear costs under unit weight Chebyshev norm is* $\max(0, -\mu^*)$.

Proof: By using similar arguments as in Theorem 3.19, it is easy to show that $-\mu^*$ is a lower bound on the optimal objective function value. Moreover, by Theorem 5.7, we know that there exists $v \in \mathcal{D}$ satisfying the ϵ-complementary slackness conditions (5.10) with $\epsilon = -\mu^*$. Thus, we define the new cost function to be

$$f_j(x_j) = \begin{cases} d_j^* x_j & \text{if } c_j^- \leq x_j \leq c_j^+ \\ \infty & \text{otherwise} \end{cases}$$

where

$$d_j^* = \begin{cases} d_j - \mu_j^* & \text{if } \tilde{x}_j < c_j^+ \text{ and } d_j - v_j < 0 \\ d_j + \mu_j^* & \text{if } \tilde{x}_j > c_j^- \text{ and } d_j - v_j > 0 \\ d_j & \text{otherwise} \end{cases} \tag{5.12}$$

Obviously, d^* is the optimum solution to the inverse primal problem with linear costs under unit weight Chebyshev norm.

■

5.4 Special Case: Generalized Minimum Cost Flows

Until now, we have assumed that the flows and potentials are conserved on the arcs of a given directed graph. However, this might not always be the case in practical applications. For example, in energy networks, nodes represent various raw materials such as crude oil and coil and various energy outputs like gas and electricity. The arcs of an energy network is, then, the transformation of one raw material into an energy output. Obviously, one cannot always expect to have a one-to-one transformation between the raw materials and the energy outputs. Hence, there exist arc multipliers, which represent the efficiency of the transformation. A network flow problem having these multipliers on the arcs is called a *generalized network flow problem.*

As mentioned already, generalized network flow problems are special cases of monotropic programs. Hence, the results that we obtained so far can be carried over to the generalized network flows with linear costs. In this section, we first provide a brief introduction to the generalized minimum cost flows and enlighten what the counterparts of the supports are in generalized flows. Then, we discuss inverse generalized minimum cost flow problem under rectilinear and Chebyshev norms.

Let $G = (N, A)$ be a digraph with a node set N of n nodes and an arc set A of m arcs. There exist capacities for the flows on arcs which we denote by $u : A \to \mathbb{R}_+^m$. The well-known linear programming formulation of a generalized minimum cost flow problem is

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij} \tag{5.13a}$$

subject to

$$\sum_{j\in N^+(i)} x_{ij} - \sum_{j\in N^-(i)} \beta_{ji} x_{ji} = b(i) \qquad \forall\, i \in N \tag{5.13b}$$

$$0 \le x_{ij} \le u_{ij} \qquad \forall\, (i,j) \in A \tag{5.13c}$$

Here, $N^+(i)$ and $N^-(i)$ are the sets of nodes adjacent from and to node i, respectively, and $\beta_{ij} > 0$ is the *multiplier* of the arc (i,j). When we send 1 unit of flow on arc (i,j), β_{ij} units of flow arrive at node j. If $\beta_{ij} < 1$, the arc is *lossy*; if $\beta_{ij} > 1$, the arc is *gainy*.

Let P be a path from node s to node t. If P^+ and P^- denote the respective sets of forward and backward arcs in P, then the *path multiplier* $\beta(P)$ is

$$\beta(P) = \frac{\prod_{(i,j)\in P^+} \beta_{ij}}{\prod_{(i,j)\in P^-} \beta_{ij}}. \tag{5.14}$$

Similarly, let C be a cycle and C^+ and C^- denote the sets of forward and backward arcs in this cycle, respectively. The *cycle multiplier* $\beta(C)$ is defined as

$$\beta(C) = \frac{\prod_{(i,j)\in C^+} \beta_{ij}}{\prod_{(i,j)\in C^-} \beta_{ij}}. \tag{5.15}$$

A cycle with multiplier $\beta(C) = 1$ is called a *breakeven* cycle.

Similar to the minimum cost flows, we can define a *residual graph* $G(\hat{x})$ for the generalized flows and *residual costs* for the arcs (i,j) of the graph G. Suppose that $\pi(i)$ is the node potential associated with node i of graph G. Then, the reduced cost of an arc (i,j) can be defined as $c^{\pi}_{ij} = c_{ij} - \pi(i) + \beta_{ij}\pi(j)$. The residual graph $G(\hat{x}) = (N, A(\hat{x}))$ with respect to a given feasible flow $\hat{x}$ consists of the arc set

$$A(\hat{x}) = (A \backslash \{(i,j) : \hat{x}_{ij} = u_{ij}\}) \cup \{(j,i) : (i,j) \in A \text{ and } \hat{x}_{ij} > 0\}.$$

The capacities, costs and multipliers of the arcs in residual graph is defined as follows.

$$u^r_{ij} = \begin{cases} u_{ij} - \hat{x}_{ij} & \text{if } (i,j) \in A \\ \beta_{ji}\hat{x}_{ji} & \text{if } (i,j) \in A(\hat{x})\backslash A \end{cases} \qquad c^r_{ij} = \begin{cases} c_{ij} & \text{if } (i,j) \in A \\ \frac{-c_{ji}}{\beta_{ji}} & \text{if } (i,j) \in A(\hat{x})\backslash A \end{cases}$$

$$\beta^r_{ij} = \begin{cases} \beta_{ij} & \text{if } (i,j) \in A \\ \frac{1}{\beta_{ji}} & \text{if } (i,j) \in A(\hat{x})\backslash A \end{cases}$$

Example 1. Consider a generalized minimum cost flow problem defined on the graph given in Figure 5.4. The residual graph corresponding to the feasible solution $\hat{x}$ with

$\hat{x}_{12} = 10$, $\hat{x}_{13} = 8$, $\hat{x}_{24} = 15$, $\hat{x}_{25} = \hat{x}_{32} = 0$, $\hat{x}_{34} = 16$, and $\hat{x}_{45} = 27$ is provided in Figure 5.5.

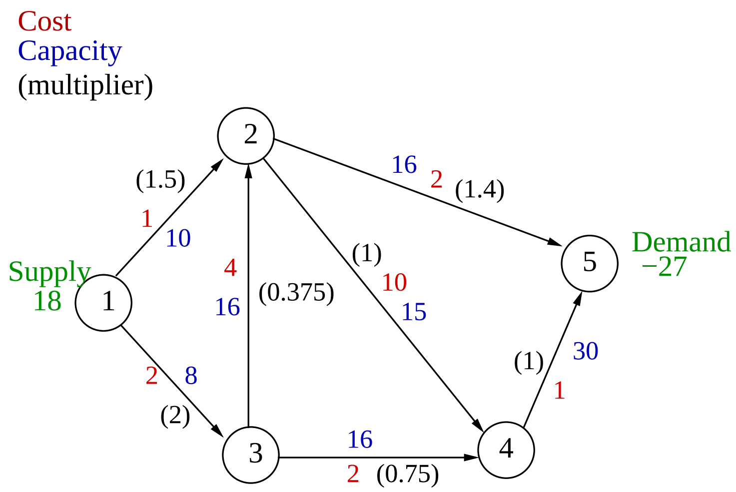

Figure 5.4: An instance of generalized minimum cost flow graph

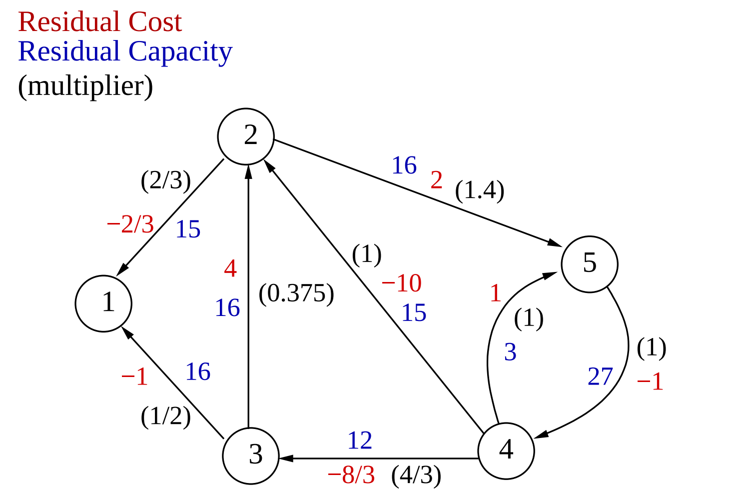

Figure 5.5: Residual graph $G(\hat{x})$ corresponding to the feasible flow $\hat{x}$

Using the residual costs and residual graph $G(\hat{x})$, we can characterize the optimality conditions for generalized flows.

Theorem 5.9. *(Ahuja* et al.*, 1993) A feasible flow x is an optimal solution of the generalized* *network flow problem if there exists node potentials* $\pi(i)$ *for all* $i \in N$ *such that*

$$c_{ij}^{\pi} \begin{cases} = 0 & \textit{if } 0 < x_{ij} < u_{ij}, \\ \geq 0 & \textit{if } x_{ij} = 0, \\ \leq 0 & \textit{if } x_{ij} = u_{ij}. \end{cases}$$

The counterparts of cycles in the generalized network flow problem are the *bicycles* (or *goggles*). According to Rockafellar (1984) there are 3 types of bicycles, which are also illustrated in Figure 5.6.

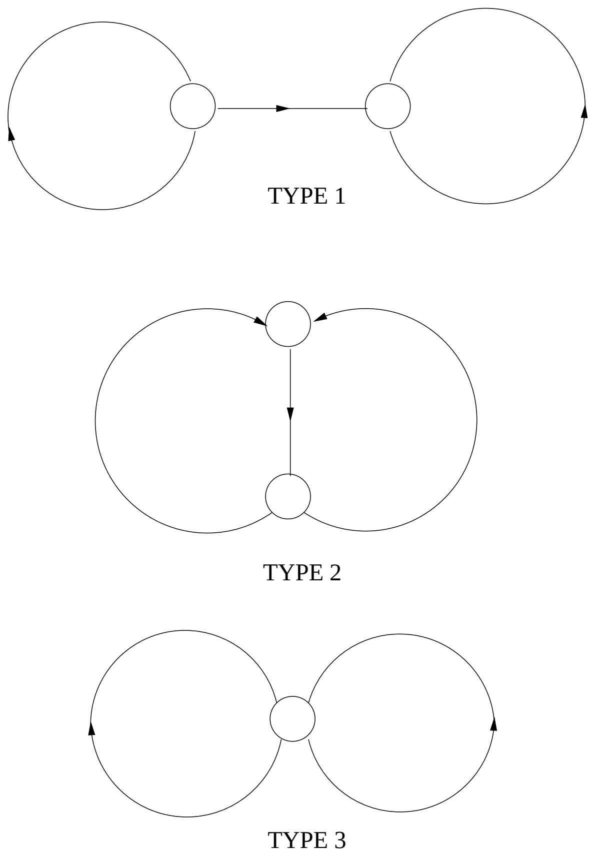

Figure 5.6: Types of bicycles (goggles)

TYPE 1. Two disjoint elementary cycles, one gainy and one lossy, together with an elementary path having only its initial node in the gainy cycle and its terminal node in the lossy cycle.

TYPE 2. Two elementary cycles, one gainy and one lossy, that have a joint portion.

TYPE 3. Two elementary cycles, one gainy and one lossy, that meet in exactly one node.

Note that TYPE 3 can be thought of as a degenerate form of the other two types.

Theorem 5.10. *A feasible flow $\hat{x}$ to a generalized minimum cost flow problem is an optimal flow if and only if the corresponding residual graph $G(\hat{x})$ does not contain any negative (cost) directed breakeven cycles or bicycles.*

Here we do not provide a proof of Theorem 5.10 and instead refer to the well-known books by Ahuja *et al.* (1993) and Rockafellar (1984).

In the given example (Example 1), the feasible solution $\hat{x}$ is not an optimum solution. Hence, according to Theorem 5.10 the residual graph in Figure 5.5 contains either a negative cost breakeven cycle or a bicycle. Indeed the feasible generalized circulation on $G(\hat{x})$ (Figure 5.5) illustrated in Figure 5.7 defines such a bicycle of TYPE 2 for Example 1.

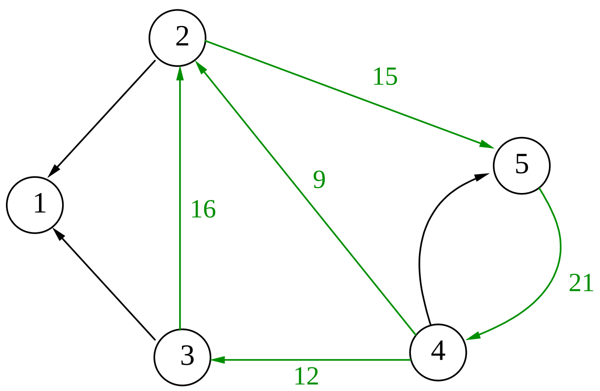

Figure 5.7: A bicycle circulation on the residual graph $G(\hat{x})$ in Example 1

Proposition 5.11. *(Rockafellar, 1984) Let G be a connected network with multipliers β that has at least one lossy or gainy cycle. Then, the elementary primary supports are the elementary breakeven cycles in G and the bicycles of Types* $1, 2,$ *and* 3*.*

Now using Proposition 5.11 and the results of the inverse monotropic programming from sections 5.2 and 5.3, we can derive the following conclusions for the (cost) inverse problems of generalized network flows under ℓ_1 and ℓ_∞ norms.

Inverse Generalized Minimum Cost Flow under Rectilinear (ℓ_1) Norm

Given an instance of generalized minimum cost flow problem and a feasible non-optimal solution $\hat{x}$ to it, the aim of the inverse problem is to perturb the cost function from c to $\hat{c}$ such that the given feasible solution is optimum with respect to the new cost function $\hat{c}$ and the total modification, $\sum_{(i,j)\in A} |c_{ij} - \hat{c}_{ij}|$, is minimized.

Here, we will define an *elementary bicycle*, denoted by $e^b \in \mathbb{R}^m$, to be the unique vector identifying a generalized circulation on a bicycle or a breakeven cycle and satisfying $|e^b(ij)| \leq 1$ for all arcs $(i,j) \in A$.

In Example 1, the elementary bicycle of the generalized circulation in Figure 5.7 is

$$e^b = \begin{pmatrix} (1,2) \\ (1,3) \\ (2,4) \\ (2,5) \\ (3,2) \\ (3,4) \\ (4,5) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -9/21 \\ 15/21 \\ 16/21 \\ -16/21 \\ -1 \end{pmatrix}.$$

Notice that the elementary bicycle e^b is written for the original graph G, not for the residual graph. Hence, the flow on arc $(3,4)$ is equal to -16.

Corollary 5.12. *Let $\mathcal{B} = \{B_1, \ldots, B_K\}$ be a minimum cost collection of disjoint elementary directed breakeven cycles and bicycles of Type $1, 2,$ or 3 in the residual graph $G(\hat{x})$. Suppose e^{b_k} denotes the elementary bicycle corresponding to the breakeven cycle or bicycle B_k in G. The objective function value of the inverse generalized minimum cost flow problem under unit weight ℓ_1-norm is*

$$-Cost(\mathcal{B}) = -\sum_{k=1}^{K} Cost(B_k) = -\sum_{k=1}^{K} \sum_{(i,j)\in e^{b_k}} c_{ij} e^{b_k}(ij). \tag{5.16}$$

Inverse Generalized Minimum Cost Flow under Chebyshev (ℓ_∞) Norm

Given an instance of generalized minimum cost flow problem and a feasible non-optimal solution $\hat{x}$ to it, the aim of the inverse problem is to perturb the cost function from c to $\hat{c}$ such that the given feasible solution is optimum with respect to the new cost function $\hat{c}$ and the maximum modification, $\max_{(i,j)\in A} |c_{ij} - \hat{c}_{ij}|$, is minimized.

Here, we will define an *elementary bicycle*, denoted by $e^b \in \mathbb{R}^m$, to be the unique vector identifying a generalized circulation on a bicycle or a breakeven cycle, denoted by B, and satisfying

$$\sum_{(i,j)\in B^+} e^b(ij) - \sum_{(i,j)\in B^-} e^b(ij) = |B|,$$

where B^+ and B^- denote the forward and backward arcs of B, respectively, and $|B|$ is the number of arcs on the bicycle or breakeven cycle B.

In Example 1, the elementary bicycle of the generalized circulation in Figure 5.7 is

$$e^b = \begin{pmatrix} (1,2) \\ (1,3) \\ (2,4) \\ (2,5) \\ (3,2) \\ (3,4) \\ (4,5) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -9/15.4 \\ 15/15.4 \\ 16/15.4 \\ -16/15.4 \\ -21/15.4 \end{pmatrix}.$$

Corollary 5.13. *Let B^* be a minimum mean cost directed breakeven cycle or a bicycle of Type $1, 2,$ or 3 in the residual graph $G(\hat{x})$ and let μ^* be its mean cost, i.e.,*

$$\mu^* = \frac{Cost(B^*)}{|B^*|} = \frac{\sum_{(i,j)\in e^{b^*}} c_{ij} e^{b^*}(ij)}{|B^*|} \tag{5.17}$$

where e^{b^*} is the elementary vector corresponding to the circulation on B^* and $|B^*|$ is the number of arcs having $e^{b^*}(ij) \neq 0$. The optimum objective function value of the inverse generalized minimum cost flow problem under unit weight Chebyshev norm is $\max(0, -\mu^*)$.

To most outsiders, modern mathematics is unknown territory. Its borders are protected by dense thickets of technical terms. Few realize that the world of modern mathematics is rich with vivid images and provocative ideas.

Ivars Peterson

6 Concluding Remarks

This chapter is dedicated to summarize the major results of this thesis and to discuss the possible further directions for future research.

6.1 Summary of Main Results

In this thesis we analyzed the inverse problems of network optimization problems and their generalizations to matroid flows and monotropic programming.

First of all, we studied the inverse maximum flow problem under Chebyshev norm. To the best of our knowledge, this problem was, in the literature, only studied under rectilinear and Hamming distances. We showed that the problem under Chebyshev norm can be solved in polynomial time by computing a maximum capacity path on the residual graph.

We also introduced a new class of inverse network flow problems which has so far not been treated in the literature. This new inverse problem models the inverse minimum cost flow problems in which a given feasible flow is made an optimum flow by only perturbing the arc flow capacities. We call this problem the capacity inverse minimum cost flow problem. We proved that under rectilinear norm the problem is $\mathcal{NP}$-hard by a reduction from the feedback arc set problem. Since the feedback arc set problem is $\mathcal{APX}$-hard and the reduction algorithm is approximation preserving, it was shown that the rectilinear problem is $\mathcal{APX}$-hard, as well. Under Chebyshev norm, the problem can be solved in strongly polynomial time by a greedy algorithm.

We also investigated a bicriteria version of the capacity inverse minimum cost flow problem. In the bicriteria problem, the number of affected arcs is minimized among the optimal solutions of the Chebyshev norm. Since this problem can be considered as a special case of the rectilinear problem, the bicriteria problem is $\mathcal{NP}$-hard, as well. Therefore, a 2-phase approximation algorithm, named Bicriteria Approximation Algorithm, was proposed to solve the bicriteria problem. The computational experiments on this algorithm show that, although the algorithm is quite fast, its performance in finding a good approximation of the optimal solution highly depends on the graph structure.

In order to achieve a well-established generalization of the inverse network flow problems under rectilinear and Chebyshev distances, we first analyzed the inverse problems of tensions on directed graphs. Since the tensions are duals of circulations, the inverse problems of maximum tension and minimum cost tension problems can be solved analogously to their flow counterparts. When an inverse problem, for instance, requires a cycle computation for the flows, then, in the tension version, this cycle computation is replaced by a cut computation. In other words, the flows and tensions preserve their duality relationship also in their inverse problems. Moreover, this duality relationship allows us to build a very intuitive generalization of the inverse network problems to the inverse matroid flows on regular matroids. It is well-known in Matroid Theory that cycles and cuts are the circuits of graphic and cographic matroids, respectively. Since graphic and cographic matroids are both regular and dual to each other, the combinatorial results of the inverse network flow and tension problems can be immediately generalized to the inverse matroid flows on regular matroids. In this case, circuits replace the cycles and cuts.

Another generalization presented in this thesis is the analysis of the cost inverse monotropic programming problems with separable linear cost functions under rectilinear and Chebyshev distances. Such monotropic programs generalize linear programs, ordinary and generalized network flow and tension problems while preserving the combinatorial properties of these problems. Hence, the combinatorial results of the cost inverse minimum cost flow and tension problems under rectilinear and Chebyshev distances can be extended to the monotropic programs, as well. A by-product of this generalization is the inverse generalized minimum cost flow problem. A generalized minimum cost flow problem is a special case of monotropic programming, which does not possess totally unimodularity. Therefore, it is an important observation of this thesis to show that the intuition used to model the inverse (ordinary) network flow problems is also valid even when the totally unimodularity is relaxed.

6.2 Future Research

In the literature there are several different versions of the inverse network flow problems that have already been analyzed and several new versions can be generated. For example, Yang *et al.* (1997) studied a version of the inverse maximum flow problem under unit weight rectilinear norm in which only upper flow capacities can be changed within a certain interval. Deaconu (2008) extended these results for the maximum flow problems with upper and lower bounds for the flow and allowed both bounds to be perturbed. The major difference between allowing both bounds to be perturbed or just one bound is the feasibility. Obviously, if, for the inverse maximum flow problem, both flow bounds can be modified, then there always exists an inverse feasible solution whereas in the latter case the inverse problem might be infeasible for some given feasible flows. However, in both cases the basic combinatorial discussions on the optimality of the inverse problem remain the same. Analogously, limiting the allowed perturbation of the problem parameters has an effect on the feasibility of the inverse problem but does not change its major combinatorial properties. Consequently, we did not consider all such versions of the inverse problems analyzed in this thesis. Depending on the practical problem in hand, one can modify the results presented in this thesis to include the necessary extra constraints. Actually, it is an important future research topic to identify the practical application areas of the inverse problems presented in this thesis and to analyze the related extensions of them depending on the given practical problem.

Recall that, in this thesis, the main emphasis was put on the rectilinear and Chebyshev norms because of their handiness. Hamming distance was also employed from time to time as a second criterion for the optimum solutions of the inverse problems under Chebyshev norm. Nevertheless, there are several other distance measures, which might be relevant for certain practical problems, and hence, should be used in the context of inverse network optimization. For instance, Sokkalingam (1995) studied the inverse minimum cost flow problem under unit weight ℓ_2-norm. However, to the best of our knowledge, there does not exist any analysis of the inverse maximum flow problem under Euclidean distance. It is a possible research direction to extend the analysis of the inverse network flow and tension problems for other distance measures such as Euclidean distance.

Another future research topic is the polyhedral analysis of the capacity inverse minimum cost flow problem under rectilinear norm. Recall that this problem is $\mathcal{NP}$-hard even for the minimum cost flow problems with unit arc flow capacities. Therefore, it is necessary to find different formulations of this problem and analyze their strength. Clearly, one possible formulation of this problem can be derived by using integer programming (IP) and LP duality. We present the ideas regarding this formulation at the end of this section. It might be also very promising to look for a quadratic programming formulation since the feedback arc set problem can be formulated as a quadratic assignment problem (Flood, 1990).

Development of efficient approximation algorithms to find good approximations of the optimal solution of the capacity inverse minimum cost flow problem under rectilinear norm is left for future research, as well. It is still an open question if there exists a polynomial time constant approximation algorithm for capacity inverse minimum cost flow problem under rectilinear norm as well as for weighted feedback arc set problem. Another open question is whether the capacity inverse minimum cost tension problem is also $\mathcal{NP}$-hard under rectilinear norm or not. We reckon that the inverse problem remains $\mathcal{NP}$-hard for the tension case. The reason for this conjecture is the fact that even if all the negative cost residual cuts could be identified in polynomial time, a set covering problem, which is known to be $\mathcal{NP}$-hard, should be solved to find the optimum solution.

A final future research topic, which should be mentioned, is the analysis of inverse monotropic programs with separable piecewise linear cost functions. As far as we know, until now, only the optimization problems with linear cost functions have been considered in the context of inverse optimization. However, not all the real-life problems can be modeled as optimization problems with linear costs. Besides, studying the inverse monotropic programs with separable piecewise linear cost functions would provide a good starting point for a research on the inverse problems of the optimization problems having more complex cost functions than linear ones.

An IP Formulation of Capacity Inverse Minimum Cost Flow Problem

In this section, we provide a mixed binary integer programming formulation of the capacity inverse minimum cost flow problem under rectilinear norm. This formulation could be a promising initial idea for a future analysis of the polyhedral structure of the problem.

We are given a digraph $G = (N, A)$ with a node set N of n nodes and an arc set A of m arcs with arc flow capacities $u : A \to \mathbb{Z}^m_+$, and a minimum cost flow problem defined on this graph. Let $c : A \to \mathbb{Z}^m$ be the arc costs. Consider the LP formulation of the minimum cost flow problem (1.1a - 1.1c) with $l_{ij} = 0$ for all $(i, j) \in A$ and its dual:

$$\max \quad \sum_{i \in N} b(i)\pi(i) \quad - \sum_{(i,j)\in A} u_{ij}\lambda_{ij} \tag{6.1a}$$

subject to

$$-\pi(i) + \pi(j) - \lambda_{ij} \leq c_{ij} \qquad \forall\, (i,j) \in A \tag{6.1b}$$

$$\pi(i) \in \mathbb{R}, \quad \lambda_{ij} \geq 0 \tag{6.1c}$$

Here, $\pi(i)$ and λ_{ij} are the dual variables corresponding to the constraints (1.1b) and (1.1c), respectively.

Let $\hat{x}$ be the given feasible flow to the minimum cost flow problem. Recall that in the rectilinear capacity inverse minimum cost flow problem we perturb the arc flow capacities from u to $\hat{u}$ such that $\hat{x}$ is an optimum solution of the minimum cost flow problem with the arc flow capacities $\hat{u}$ and the total change $\sum_{(i,j)\in A} |\hat{u}_{ij} - u_{ij}|$ is minimum. Some observations on the properties of the minimum cost flow problem with the new arc capacities $\hat{u}$ are as follows:

1. Since cost vector c is not changed and $\hat{x}$ is an optimum solution, the optimum objective function value is

$$\hat{z} = \sum_{(i,j)\in A} c_{ij}\hat{x}_{ij} = \sum_{i\in N} b(i)\hat{\pi}(i) - \sum_{(i,j)\in A} \hat{u}_{ij}\hat{\lambda}_{ij} \tag{6.2}$$

   where $\hat{u}$ is the new capacity vector for the arc set A and $\hat{\pi}$ and $\hat{\lambda}$ are the optimum dual variables corresponding to the minimum cost flow problem with the arc capacities $\hat{u}$.

2. Let us define

$$\begin{aligned} \hat{X}_1 &= \{(i,j) \in A : \hat{x}_{ij} = u_{ij}\}, && (6.3a)\\ \hat{X}_2 &= \{(i,j) \in A : 0 < \hat{x}_{ij} < u_{ij}\}, && (6.3b)\\ \hat{X}_3 &= \{(i,j) \in A : \hat{x}_{ij} = 0\}. && (6.3c) \end{aligned}$$

   (i) Since the arcs in set $\hat{X}_1$ have flows at the upper capacity, the capacities of these arcs remain the same, i.e. $\hat{u}_{ij} = u_{ij}$ for all $(i,j) \in \hat{X}_1$.

   (ii) For the arcs in sets $\hat{X}_2$ and $\hat{X}_3$, if $\hat{u}_{ij} = u_{ij}$ then by complementary slackness theorem for linear programs $\hat{\lambda}_{ij} = 0$ (Hamacher and Klamroth, 2001).

   (iii) We know that if the capacity of an arc is changed, i.e. $\hat{u}_{ij} \neq u_{ij}$, then it is set to the flow value $\hat{x}_{ij}$ (Proposition 2.11). Using this fact and the complementary slackness theorem, we can draw the following conclusion:

$$\hat{u}_{ij}\hat{\lambda}_{ij} = 0 \quad \forall\ (i,j) \in \hat{X}_3 \tag{6.4}$$

   It is very easy to prove this claim. If the capacity of the arc is not changed, i.e. $\hat{u}_{ij} = u_{ij}$, then $\hat{x}_{ij} = 0 < \hat{u}_{ij}$. Hence, $\hat{\lambda}_{ij} = 0$ by complementary slackness. If the capacity is changed, then $\hat{u}_{ij} = 0$. Thus, the equality (6.4) holds.

Let us define the variables of the mixed binary integer programming formulation.

- $\hat{u}_{ij} \geq 0$ are the new capacity values for the arcs $(i,j) \in A$.
- $\pi(i) \in \mathbb{R}$ and $\lambda_{ij} \geq 0$ are the dual variables for all $i \in N$ and $(i,j) \in A$, respectively.
- For every arc $(i,j) \in \hat{X}_2 \cup \hat{X}_3$,

$$y_{ij} = \begin{cases} 1 & \text{if} \quad \hat{u}_{ij} = \hat{x}_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the formulation of the capacity inverse min cost flow problem under rectilinear norm is

$$\min \sum_{(i,j)\in \hat{X}_2 \cup \hat{X}_3} (u_{ij} - \hat{u}_{ij}) \tag{6.5a}$$

subject to

$$\sum_{i\in N} b(i)\pi(i) - \sum_{(i,j)\in \hat{X}_1} u_{ij}\lambda_{ij} - \sum_{(i,j)\in \hat{X}_2} \hat{x}_{ij}\lambda_{ij} = \hat{z} \tag{6.5b}$$

$$-\pi(i) + \pi(j) - \lambda_{ij} = c_{ij} \qquad \forall\ (i,j) \in \hat{X}_1 \cup \hat{X}_2 \tag{6.5c}$$

$$-\pi(i) + \pi(j) - \lambda_{ij} \leq c_{ij} \qquad \forall\ (i,j) \in \hat{X}_3 \tag{6.5d}$$

$$\lambda_{ij} \leq My_{ij} \qquad \forall\ (i,j) \in \hat{X}_2 \cup \hat{X}_3 \tag{6.5e}$$

$$\hat{u}_{ij} \leq u_{ij} - (u_{ij} - \hat{x}_{ij})y_{ij} \qquad \forall\ (i,j) \in \hat{X}_2 \tag{6.5f}$$

$$\hat{u}_{ij} \leq u_{ij}(1 - y_{ij}) \qquad \forall\ (i,j) \in \hat{X}_3 \tag{6.5g}$$

$$\hat{u}_{ij} \geq 0 \quad \pi(i) \in \mathbb{R} \quad \lambda_{ij} \geq 0 \quad y_{ij} \in \mathbb{B} \tag{6.5h}$$

where $\hat{x}$, $\hat{z}$, c, b and u are part of the given data and M is a sufficiently large number. In this formulation, the first equality (6.5b) is derived by the optimality of $\hat{x}$ for the minimum cost flow problem with a modified arc capacity vector and the duality. The inequalities (6.5c) and (6.5d) follow from the duality theory whereas (6.5e) holds by (ii). The inequalities (6.5f) and (6.5g) determine the upper bound values for the new capacities. Since the objective function is the minimization of $-\hat{u}$, the vector $\hat{u}$ always takes the upper bound value, which is actually identified via the binary variables y.

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List of Tables

List of Figures

Index

Declaration

I hereby declare that I am the only author of this work and that no other sources than those listed have been used. The following sections of this thesis are substantially based upon work published or submitted for publication elsewhere, or include collaborative work:

1. Section 2.2 is based upon "Capacity Inverse Minimum Cost Flow Problem", by Ç. Güler and H.W. Hamacher, accepted for publication in "Journal of Combinatorial Optimization" and online since 30 April 2008.

2. Parts of Chapter 3 and Chapter 5 are based upon "Inverse Tension Problems and Monotropic Optimization", by Ç. Güler, submitted to "Journal of Combinatorial Optimization".

Kaiserslautern, February 2009

Çiğdem Güler

Curriculum Vitae

Çiğdem Güler

Personal Data

Birthdate	:	September 1^{th}, 1981
Birthplace	:	Polatli, Turkey
Nationaliy	:	Turkish

Education

Aug 2006 - Feb 2009	Ph.D. student at Dept. of Mathematics, University of Kaiserslautern, Germany **Thesis:** On Inverse Network Problems and their Generalizations **Supervisor:** Prof. Dr. Horst W. Hamacher
Oct 2004 - Jul 2006	M.Sc at Department of Mathematics, University of Kaiserslautern, Germany (scholarship from Deutscher Akademischer Austauschdienst (DAAD) and Türk Eğitim Vakfi) **Final Grade:** 1,0 (very good) **Thesis:** Radiation Therapy Planning - Combining Intensity and Realization Problems **Supervisor:** Prof. Dr. Horst W. Hamacher
Sep 1999 - Jun 2004	B.Sc at Industrial Engineering, Middle East Technical University (METU), Turkey **Final Grade:** 3,75 (4,00 is the best possible grade.)

Work Experience

Oct 2008 - Feb 2009	Teaching assistant for Integer Programming course at Dept. of Mathematics, University of Kaiserslautern
May 2008 - Jul 2008	Project assistant in Optimization Group at Dept. of Mathematics, University of Kaiserslautern
Oct 2006 - Feb 2007	Teaching assistant for Integer Programming course at Dept. of Mathematics, University of Kaiserslautern
Aug 2005 - Apr 2006	Scientific assistant in Optimization Group at Dept. of Mathematics, University of Kaiserslautern
Jan 2005 - Jun 2005	Scientific assistant at Dept. of Image Processing, ITWM, Kaiserslautern